Numerical Pricing & Risk Management of Energy Commodities Derivatives

Master Thesis
M.Sc. Computer Simulation in Science

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Contents

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Erklärung

Ich versichere, dass meine Masterarbeit selbständig verfasst wurde sowie keine anderen als die angegebenen Quellen und Hilfsmittel benutzt sowie Zitate kenntlich gemacht wurden.

Wuppertal, den 28 Aug 2013

Konstantinos A. Sofos

Part I.

Introduction

1. Risk Management & Price Modeling

Risk management and price modeling is important in the energy industries because of the volatility of energy products prices. Price volatility can reduce the profit of business strategies and affect consumers. For this reason managing price risk has become a necessity in the energy industries to maintain profitability and to avoid a competitive disadvantage. The use of financial derivatives, both traded in exchanges and over-the-counter (OTC), has developed as a low cost method of hedging price risk. A wide variety of derivative contracts exist, including options, forwards and futures which can be put together to achieve a wide variety of objectives.

Risk is defined as a situation in which a variable is likely to take on a value differing from that which was expected. Price risk refers to exposure to adverse price moves in the cash market. A producer or supplier loses if prices move lower in the cash market. The consumer loses if prices move higher. Basis risk refers to the difference between the price used as a benchmark in a transaction and the price for the actual goods changing hands. If the difference between the benchmark price and the actual price does not remain constant, there will be a loss or a gain on one side of the deal. Economists and other analysts use a statistical tool, the *standard deviation*, to measure risk. The standard deviation measures the spread of possible outcomes around the average, or expected, value of the variable in question.

Larger values of the standard deviation imply more risk. Although managing price risk has become a major consideration for energy companies, this doesn't necessarily mean using derivatives. Several alternatives to derivatives exist that might accomplish similar results. Vertical integration, the incorporation of the various stages of the production process, from exploration and production to final retail distribution, into one entity, allows the firm to control price risk. Vertically integrated firms are able to manage how a change in the price of a primary factor of production is incorporated into the cost structure of the firm.

Since volatile energy prices are not likely to stabilize in the future, firms must undertake a strategy to protect themselves from price volatility. The strategy chosen must be cost effective, flexible, and reliable. Financial derivatives fit these requirements for many firms in the energy industries

1.1. Introduction To Energy Derivatives

For many market participants, energy derivatives appear to be a new phenomenon. Although it is true that traded derivatives are a relatively new concept in the energy markets, the structures have been around for centuries and contracts with derivative characteristics have existed in energy markets for decades.

For example, a futures contract on 1000 barrels of light, sweet, crude oil dated July, 2013, at a price of $90 obligates the owner of that contract to purchase oil at that time, at those terms. In what sense does the futures contract have value? If, near the settlement date in July 2013, light, sweet crude is selling for $100 per barrel on the spot market, holding a legally enforceable right to buy the oil at $90 per barrel creates a value of $10 per barrel for the owner of the futures contract. Conversely, if oil is available on the spot market for $80 per barrel on the July, 2013 settlement date, the futures contract is a liability for the contract holder in that it requires the oil buyer to pay $10 more for oil than the market price.

Derivatives traded on the energy exchanges are liquid[1], while OTC contracts generally are not. A party on either side of an exchange-traded contract can cancel its position at any time by buying or selling a contract that is opposite its original contract. For example, if a firm had purchased a futures contract on 1000 barrels of crude oil it could sell a contract with identical terms which would effectively cancel the firm's obligation. From the point of view of the exchange, the firm would have netted out its position, having bought and sold contracts obligating it to 1000 barrels of oil, leaving it, in effect, out of the market. This type of transaction can be undertaken at any time because all contracts are standardized and have the central clearinghouse, which is owned by the market, as counterpart to the contracts. If the contract were OTC, the only way the contract could be terminated or modified would be through mutual negotiation and agreement between the principals. A firm that chose to abrogate an OTC contract it found financially disadvantageous would likely have to pay penalties to the counterpart, who would suffer damages. If the terms of an OTC contract are such that one of the principals to the agreement is suffering large losses, that party might not be able to meet the terms of the agreement, raising the possibility of default. The costs of default can be substantial and are very real since OTC contracts are legally enforceable contracts.

Trades on organized exchanges are anonymously, cost efficiently, and competitively implemented with instantaneous price transparency. This is helpful to traders who might want to put a business strategy in place cheaply, quickly, and without revealing their strategy to other market participants. OTC contracts are, in effect, the opposite. Since they are one-on-one arrangements, the principals to the agreement are closely related to one another.

[1] Market liquidity is an asset's ability to be sold without causing a significant movement in the price and with minimum loss of value.

Figure 1.1.: The setting of a typical trading floor schematically depicted. Left-side figure shows the typical products on the trading floor. Right-side figure displays who is the problem owner and where does it fit in.

1.1.1. Derivatives Contract Types

There are three main derivatives contract types.

1. **Options**. An option contract gives the owner the right, but not the obligation, to buy or sell quantities of the underlying asset at a fixed price known as the strike price. Option based strategies allow the owner to participate in favorable outcomes while minimizing the effect of negative outcomes. Offsetting this favorable result, an options based strategy is more expensive than futures based strategies.

2. **Futures**. A future contract is a contractual agreement, generally made on the trading floor of a futures exchange, to buy or sell a particular commodity or financial instrument at a pre-determined price in the future. Futures contracts detail the quality and quantity of the underlying asset; they are standardized to facilitate trading on a futures exchange. Some futures contracts may call for physical delivery of the asset, while others are settled in cash. One of the main attractions of futures contracts is the virtual elimination of counterpart credit risk, because the financial performance and commodity delivery are guaranteed by the exchange. Another frequently mentioned benefit of futures is the reduction of transaction costs due to contract standardization. Finally, because they are settled daily, computing the mark-to-market[2] (MTM) value of futures contracts does not require discounting, making them much simpler to evaluate than their close cousins: forwards

3. **Forwards**. A forward contract is a cash market transaction (cash-and-carry agreement) in which delivery of the commodity is deferred until after the contract has been made. Although the delivery is made in the future, the price is determined on the initial trade date. Unlike exchange-traded futures,

[2]Mark-to-market valuation refers to accounting for the "fair value" of an asset or liability based on the current market price.

forward contracts are over-the-counter (OTC) products. They need not be standardized, and they can be structured in the way most convenient to the counterparties. This flexibility is one of the reasons why forward contracts are currently more popular than futures. Furthermore forward contracts, unlike futures, are not settled daily. On the positive side, this means that a contract holder does not have to worry about having daily access to cash to satisfy margin requirements. On the negative side, if the market moves in the right direction (i.e., the contract ends up being in-the-money (ITM)), the contract holder becomes exposed to counterpart credit risk. A call option is ITM, when the option's strike price is below the market price of the underlying asset. A put option is ITM, when the strike price is above the market price of the underlying asset. Being ITM does not mean you will profit, it just means the option is worth exercising. This is because the option costs money to buy.

1.2. Definition of Long and Short Position

By going long a call, the option holder has the rights to buy the underlying at the strike price before or on expiry date. The maximum risk to this buyer, if he goes long one option contract is the option premium. The maximum profit he can make is unlimited and is derived from the difference between the market rate of the underlying at the time of exercise and the strike price minus the premium. By going short a call, one is basically writing with intent to sell it in the open market of course. The option seller has the obligation to sell the underlying at the strike price if the option holder exercises the option. The maximum risk for the option seller is unlimited and is derived from the difference between the current market price of the underlying at assignment and the strike price minus the option premium. The maximum profit the option seller can reap is the premium.

By going long a put, the option holder has the right to sell the underlying at the strike price before or on the expiry date. The maximum risk to the buyer is the premium as like the long call trade. The maximum profit is unlimited and is the difference between the strike price and the market price of the underlying when the option is exercised. By shorting a put, the option seller writes a put option and has the obligation to buy the underlying at strike price if the option holder exercises his put option. The maximum risk for the option seller is unlimited and is derived from the difference between the strike price and the market price of the underlying at assignment minue the premium. The maximum profit for the option seller is the premium.

1.3. Principles Of Hedging And Price Discovery

An important application of derivatives is hedging. Hedging means to eliminate or limit risks. For example, consider a trader who holds (is "long") a call option of Light Sweet Crude Oil - WTI (1,000 barrels) and wants protection against a possible decline of the price below a value **K** in the next three months. The trader could buy put options on this stock with strike **K** and a maturity that matches his three months time horizon. Since the trader can exercise his puts when the share price falls below **K**, it is guaranteed that the stock can be sold at least for the price **K** during the life time of the option. With this strategy the value of the stock is protected. The premium paid when purchasing the put option plays the role of an insurance premium. — Hedging is intrinsic for calls. The writer of a call must hedge his position to avoid being hit by rising asset prices.The principle behind establishing equal and opposite positions in the cash and futures or options markets is that a loss in one market should be offset by a gain in the other market.

1.3.1. Fair Value

What would be the fair price of the forward contract if we did not or could not hedge? Clearly, it would be an expectation at time t of the forward spot prices S_T.

$$F_{value} = \mathbb{E}_t(S_T) = S_0 e^{rt} \tag{1.3.1}$$

or in the case that we have also a storage cost U (e.g. storage cost of natural gas) the fair value will be given by

$$F_{value} = \mathbb{E}_t(S_T) = (S_0 + U)e^{rt} \tag{1.3.2}$$

where U is the the storage cost of the commodity at $t = 0$ and over the life of a forward contract. If some or all of the cost is not spent at $t = 0$ then this future cost should be discounted at the risk-free rate.

What kind of principle is so powerful to serve as basis for a fair valuation of derivatives? The concept is arbitrage, or rather the assumption that arbitrage is not possible in an idealized market. Arbitrage means the existence of a portfolio, which requires no investment initially, and which with guarantee makes no loss but very likely a gain at maturity. Or shorter: arbitrage is a self-financing trading strategy with zero initial value and positive terminal value.

If an arbitrage profit becomes known, arbitrageurs will take advantage and try to lock in. This makes the arbitrage profits shrink. In an idealized market, informations spread rapidly and arbitrage opportunites become apparent. So arbitrage cannot last for long. Hence, in efficient markets at most very small arbitrage opportunities are observed in practice. For the modeling of financial markets this leads to postulate the

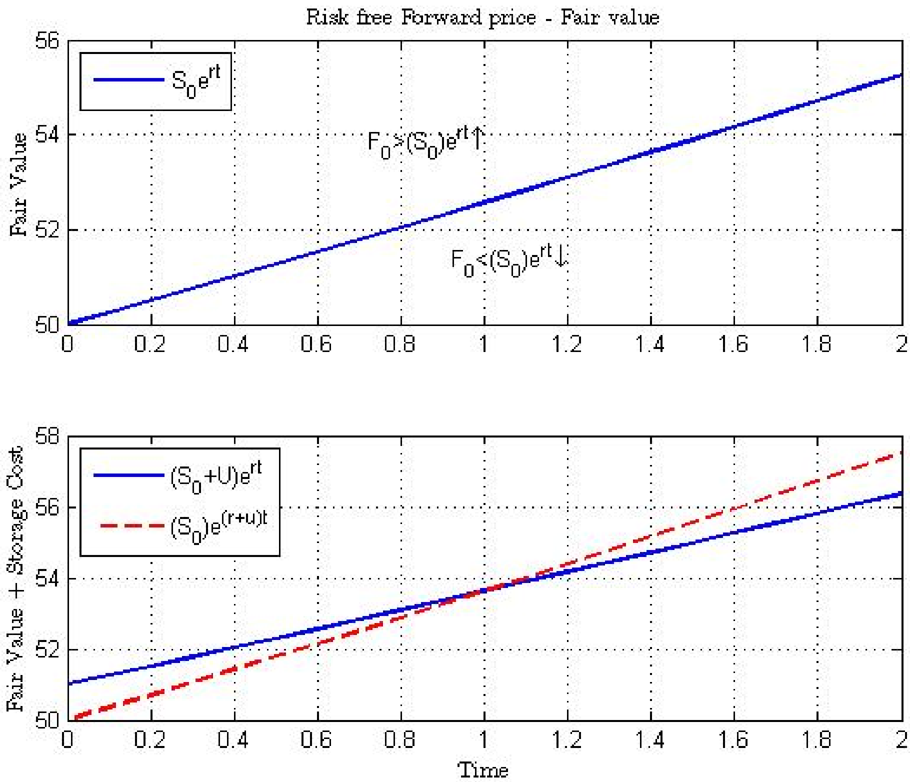


Figure 1.2.: The fair value of a forward contract as function of time to maturity (up) without and (down) with two different types of storage cost. $S_0 = 50$, $T = 2$ years and storage cost $U = 1$

no-arbitrage principle: One assumes an idealized market such that arbitrage is ruled out [see 3.1]. Arguments based on the no-arbitrage principle resemble indirect proofs in mathematics: Suppose a certain financial situation. If this assumed scenario enables constructing an arbitrage opportunity, then there is a conflict to the no-arbitrage principle. Consequently, the assumed scenario is impossible in practice.

2. Standard Energy Options

Standard energy options, such as calls and puts, are some of the most frequently used risk management tools. The literature on options is quite extensive (e.g., see McMillan, 1992; Hull, 1999; Cox and Rubinstein, 1985). Therefore, we are not going to spend much time on their analysis. We remind the reader that in energy markets a call option is the right, but not the obligation, to buy energy at a predetermined strike price, and a put option is the right, but not the obligation, to sell energy at a predetermined strike price. European-style options are exercised only once, at the specified exercise day, while American options can be exercised any time before the exercise date.

Thus, by definition, there is not much difference between calls and puts in energy markets and calls and puts in all other markets. What sets them apart is an unusual diversity of traded energy options, a natural consequence of the diversity in the underlying commodity, especially power. Typically, energy option specifications include:

- Location
- Exercise time
- Delivery conditions, for example, in the case of power, the type of delivered power (on-peak, off-peak, round-the-clock)
- Strike (exercise price)
- Volume

2.1. Energy Options Classes

2.1.1. Financially Settled Options

For options to be settled financially, there must be a widely accepted financial index against which the options are exercised. In the gas market the financial settlement of gas options is common.

The payoff of a financially settled European call and put on energy is not different from any other financially settled call and put. If K is the strike price, and S is the price of the index against which the option is settled, the call's and put's pay-off is given by

$$C(T) = max(0, S_T - K) = (S_T - K)^+, \\ P(T) = max(0, K - S_T) = (K - S_T)^+. \tag{2.1.1}$$

2.1.2. Physically Settled Options

Physical settlement is particularly relevant for power options, since the power markets have not yet developed a solid financial index for financial settlement.

Consider, for example, a monthly call option. Physical settlement means that the option holder has the right, but not the obligation, to buy, paying the strike price K, the commodity for the period of one month. Buying the commodity is only the first step in extracting the value from the option. To realize the option payoff, the option owner must now sell the commodity at the spot market. The option is then effectively an option on the average spot price inside the contractual month.

We can try to proxy for this average in two ways. In one approach we use a convergence argument and represent the average spot price as a value to which monthly forward prices converge at expiration. The advantage of this approach is that forward prices in most energy markets are quoted and reliable. The disadvantage of this approach is that the time-basis between the settlement of the forward price (at the beginning of the contract month) and the time the average settles (end of the contract month) can be very significant in power markets (as high as 40 €/MWh). This disqualifies the use of forward prices as a proxy in power option pricing. Incidentally, for forward contract pricing, whether settled physically or financially, the use of the forward price is not a problem as the forward price at expiration and the average spot price are statistically (on average) equal. However, for option pricing statistical equality is not enough. The above consideration is very important for power markets. In natural gas markets, the use of forward prices as a proxy for value seems to work well.

In the second approach, we use an average of daily spot prices over the delivery month. Again, the daily spot prices are rather transparent in most markets. This approach corresponds to taking the commodity into the month and trading it on a daily basis at daily spot prices. The justification for this approach is based on the assumption that the monthly average of daily prices should be close to the monthly spot index (in natural gas markets). In reality, the average of daily prices is a random variable. Its relation to another random variable, the monthly spot index, and their joint behavior must be thoroughly studied and properly modeled.

Once we have chosen the method of representing the price **S** at which we can sell the commodity at the spot market, the option payoff is the same as for the financially settled options.

2.1.3. Daily and Index or Cash Option

These is another popular group of options—options on the spot commodity. A daily option is exercised every day during the exercise month. It allows its owner to make daily decisions during the exercise month about buying (call option) or selling (put option) spot gas or power at a fixed strike price. In index or cash option, the option is exercised every day during the exercise month with a specified monthly index as a strike price. It allows its owner to make daily decisions during the exercise month about buying (call option) or selling (put option) spot gas or power at a strike price determined at the beginning of the month as a settled value of the monthly index.

Options on the spot commodity are very common among energy derivatives, because they answer the real need to manage price risks on a daily basis. They are typically structured as a strip of options exercised daily during a certain time period (month, quarter, season, and so on). Therefore, their payoffs can be represented as

$$\begin{aligned} C_{daily} &= \sum_{i=1}^{n} max(0, S_T - K) = \sum_{i=1}^{n} (S_T - K)^+ \\ P_{daily} &= \sum_{i=1}^{n} max(0, K - S_T) = \sum_{i=1}^{n} (K - S_T)^+ \end{aligned} \tag{2.1.2}$$

with n the days of the exercised period.

Daily and especially cash options in energy markets have a very different character from similar option structures in financial markets. They are not derivative products as such, since the underlying of those options is not traded directly. This is especially pertinent in power markets where the spot commodity itself does not exist and unlike other markets power purchased on one day cannot be resold on another.

Gas options are typically financially settled against a specified financial spot price index, such as the Gas Daily Index. Power daily options are typically physically settled and therefore the variable S in the above payoffs represents the spot price in the physical power markets.

2.2. Spread Options

The significance of spread options in the energy markets is very big. Practically every energy asset and every structured deal has a spread option embedded in it. Typically, it is a call or a put option with the exception that the underlying is now a two-commodity portfolio, instead of a single contract

$$\begin{aligned} C = max[0, (S_{power} - S_{fuel}) - K) = [(S_{power} - S_{fuel}) - K]^+, \\ P = max[0, K - (S_{power} - S_{fuel})) = [K - (S_{power} - S_{fuel})]^+. \end{aligned} \tag{2.2.1}$$

As a trading tool, they are used to stabilize operational cash flows, to mitigate geographical and calendar risks, and to arbitrage market inefficiencies.

2.2.1. Spark Spread

Spark spread is the difference between the price of electricity (output) and the prices of its primary fuels (inputs). Primary fuels are natural gas, coal, residual fuel oil, and uranium. The spark spread between electricity and natural gas is the most common. Spark spreads are traded over-the-counter (OTC).

The spark spread can be used to financially replicate the physical reality of a power plant to replicate a short position in fuels and a long position in electricity. Power plants can be considered as European call options

2.3. Strategies With Options

Traders often combine long and short options with different strike price in order to create option strategies:

- **Straddle**: this strategy is used when a high volatility is expected on the price of the underlying but no directional information is available, i.e. it is unknown whether it will be an upside or a downside volatility. The components are just a put and a call with the same strike price.
- **Strangle**: a major movement is expected with uncertainty on the direction. The components are a long put with strike K_1 and a long call with a strike price K_2 , with $K_1 < K_2$.
- **Butterfly**: is used when it is expected that the price of the underlying will remain in the vicinity of K_2 . The components are: a long call with strike K_1, two short calls with strike K_2 and one long call with strikeK_3 , with $K_1 < K_2 < K_3$.
- **Condor**: similar to a butterfly but with a wider range which allows to contain slightly higher volatilities than the butterfly. The components of the condor are: long call with strike K_1 , short call with strike K_2 , short call with strike K_3 and a long call with strike K_4 , with $K_1 < K_2 < K_3 < K_4$.

Part II.

Option Pricing Models & Hedging Strategies

3. Modeling Tools for Option Pricing

3.1. Main Assumption: No Arbitrage

The assumption that there is no arbitrage is used in quantitative finance to calculate a unique risk neutral price for derivatives. When the actual futures price does not equal the theoretical futures price, arbitrage profits may be made.

Consider an asset price S_t. Suppose we divide the time period [t, T] into small intervals of equal size Δ. For each time $t + i\Delta$ with $i = 1, \ldots, n$ we observe a different $S_{t+i\Delta}$. The $S_{t+\Delta} - S - t$ is the change in asset price at time t. Choose a working probability from all available synthetic probabilities, and denote it by $\mathbb{P}^*$.

Then, we can always calculate the expected value of this change under this probability. In the case of $\mathbb{P}^* = \tilde{\mathbb{P}}$, we obtain the risk-neutral expected net return by

$$E_t^{\tilde{\mathbb{P}}}[S_{t+\Delta} - S_t] = r_t S_t \Delta. \tag{3.1.1}$$

Now we can use the probability switching method and exploit the Martingale property [1]. For example, for the risk-neutral probability we have

$$[S_{t+\Delta} - S_t] = r_t S_t \Delta + \epsilon_t. \tag{3.1.2}$$

Now the error term ϵ_t can be written in the equivalent form

$$\epsilon_t = \sigma(S_t) S_t \Delta W_t, \tag{3.1.3}$$

where ΔW_t is a Wiener process increment with variance equal to Δ. Thus, the arbitrage-free dynamics under the $\tilde{\mathbb{P}}$ measure can be written as

$$[S_{t+\Delta} - S_t] = r_t S_t \Delta + \sigma(S_t) S_t \Delta W_t \tag{3.1.4}$$

Letting $\Delta \to 0$, this equation becomes a stochastic differential equation (SDE), that represents the arbitrage-free dynamics under the synthetic probability,$\tilde{\mathbb{P}}$, during an infinitesimally short period dt. Symbolically, the SDE is written as

$$dS_t = r_t S_t dt + \sigma_t S_t dW_t \tag{3.1.5}$$

The dS_t and dW_t represent changes in the relevant variables during an infinitesimal time interval. Given the values for the (percentage) volatility parameter,$\sigma(S_t)$, these equations can be used to generate arbitrage-free trajectories for the S_t.

3.2. Option Pricing Models

The classical Black-Scholes equation to price an option on an asset is given by [BS73]

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S} r S_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S_t^2 = rV, \qquad (3.2.1)$$

with $t \in [0, \infty)$ and $S \in [0, S_{max}]$. The Black-Scholes (BS) equation can be derived by several different approaches [FRouah] [Seydel].

It is well known that the Black-Scholes assumptions are not very realistic. A known issue with the Black-Scholes model is the appearance of a smile in the implied option's volatility. In particular for a given expiration, options whose strike price differs substantially from the underlying asset's price commands higher prices (and thus implied volatilities) than what is suggested by standard option pricing models. But yet, the Black-Scholes formula is routinely used by options traders, although these traders know better than anybody else that the assumptions behind the model are problematic. So, how can a trader still use the Black-Scholes formula if the realized volatility is known to fluctuate significantly during the life of the option? This question needs to be carefully considered. In the end, we will see that there really are no inconsistencies in traders' behavior. We can explain this as follows:

1. First, note that the Black-Scholes formula is simple and depends on a small number of parameters. In fact, the only major parameter that it depends on is the volatility, σ. A simple formula has some advantages. It is easy to understand and remember. But, more importantly, it is also easy to realize where or when it may go wrong. A simple formula permits developing ways to correct for any inaccuraciesinformallyby making subjective adjustments during trading.

2. An important aspect of the Black-Scholes formula is that it has become a convention. In other words, it has become a standard among professionals and also in computer platforms. The formula provides a way to connect a volatility quote to a dollar value attached to this quote. This way traders use the same formula to put a dollar value on a volatility number quoted by the market. This helps in developing common platforms for hedging, risk managing, and trading volatility.

3. Thus, once we accept that the use of the Black-Scholes formula amounts to a convention, and that traders differ in their selection of the value of the parameter σ, then the critical process is no longer the option price, but the

volatility. This is one reason why in many markets, such as caps, floors, and swaptions markets, the volatility is quoted directly.

One way to account for the imperfections of the Black-Scholes assumptions would be for traders to adjust the volatility parameter. However, the convention creates new risks. Once the underlying is the volatility process, another issue emerges. For example, traders could add arisk premiumto quoted volatilities. Just like the risk premium contained in asset prices, the quotes on volatility may incorporate a risk premium.

The volatility smile and its generalization, the volatility surface, could then contain a great deal of information concerning the implied volatilities and any arbitrage relations between them. Trading, pricing, hedging, and arbitraging of the smile thus become important.

An alternative to addressing the implied volatility smile or smirk is the family of stochastic volatility models in which the volatility is itself driven by a second stochastic differential equation (higher-dimensional models[1]). Along the line, Heston (1993) proposed a mean-reverting square root variance process that is correlated with the stock movement. A major advantage of the Heston[2] model is that semianalytical equations for the option value are available via Fourier inversion of a characteristic function of the Heston process.

3.2.1. The Black-Scholes Model

3.2.1.1. Derivation of the Black-Scholes Equation

The derivation of the basic Black-Scholes options pricing equation follows from imposing the condition that a riskless portfolio made up of stock and options must return the same interest rate as other riskless assets, assuming stock and options prices are in a market equilibrium. The portfolio will have an options component and a variable quantity of stock so that it remains riskless at all positive stock prices. That leads to a relationship between the option price. The derivation takes a fixed risk-free interest rate r and the stock's volatility σ to be known constants through time T when any options will expire. It is assumed that there are no transactions costs or constraints, no taxes, no dividends, and no liquidity constraints. Trading in securities is in continuous units and instantaneous; price changes are completely unaffected by the trader under consideration. It is assumed that no risk-free arbitrage opportunities exist.

[1]Except from Heston model there are several other models like Hull-White Model. The crucial difference between the Heston model and the Hull-White model is the assumption of nonzero correlation between the price and volatility process.

[2]The model's popularity arises less from its empirical performance and more from its relative tractability.

Assume asset price evolve according to the stochastic process called geometric Brownian motion.

$$dS_t = \mu dt + \sigma dW_t, \quad t \in [0, \infty) \tag{3.2.2}$$

Where μ is the drift term, σ the volatility and both are constants. The lognormal evolution follows

$$S(t) = S_0 exp((\mu - \frac{\sigma^2}{2})t + \sigma dW_t) = S_0 exp((\mu - \frac{\sigma^2}{2})t + \sigma\epsilon\sqrt{T}) \tag{3.2.3}$$

Substituting the risk-neutral lognormal asset price path into the expectation in results in the integrated expectation as given by

$$c = e^{-rT}\mathbb{E}^T[max(S_T - K, 0)], \tag{3.2.4}$$

$$c = e^{-rT}\int_{-\infty}^{\infty} max(0, S_0 exp((\mu - \frac{\sigma^2}{2})t + \sigma\epsilon\sqrt{T}) - K)\frac{1}{\sqrt{2\pi}}e^{-\epsilon^2/2}d\epsilon, \tag{3.2.5}$$

where ϵ is normally distributed. The max term is eliminated by integrating only over the limits that yield a positive value

$$S_0 e^{\left(r-\sigma^2/2\right)T+\sigma\epsilon\sqrt{T}} - K \geq 0, \tag{3.2.6}$$

which occurs when [Din2005]

$$\epsilon_1 \geq \frac{1}{\sigma\sqrt{T}}\left[ln\left(\frac{K}{S_0}\right) - \left(r - \frac{\sigma^2}{2}\right)T\right]. \tag{3.2.7}$$

Changing to the releveant limits of integration gives

$$c = \frac{e^{-rT}}{\sqrt{2\pi}}\int_{\epsilon_1}^{\infty}\left(S_0 e^{\left(r-\sigma^2/2\right)T+\sigma\epsilon\sqrt{T}} - K\right)e^{-\epsilon^2/2}d\epsilon. \tag{3.2.8}$$

Splitting the integral allows the elimination of r from the first term

$$c = \frac{S_0}{\sqrt{2\pi}}\int_{\epsilon_1}^{\infty}\left(e^{-\epsilon^2/2+\sigma\epsilon\sqrt{T}-\left(\sigma^2/2\right)T}\right)d\epsilon - \frac{Ke^{-rT}}{\sqrt{2\pi}}\int_{\epsilon_1}^{\infty}e^{-\epsilon^2/2}d\epsilon. \tag{3.2.9}$$

The exponential in the first term is simplified as square

$$c = S_0\frac{1}{\sqrt{2\pi}}\int_{\epsilon_1}^{\infty}e^{-\frac{1}{2}\left(\epsilon-\sigma\sqrt{T}\right)^2}d\epsilon - Ke^{-rT}\frac{1}{\sqrt{2\pi}}\int_{\epsilon_1}^{\infty}e^{-\frac{\epsilon^2}{2}}d\epsilon. \tag{3.2.10}$$

The integral of a normal probability distribution in the cumulative normal distribution function,

$$\Phi(x_1) = \int_{-\infty}^{x_1} e^{-x^2/2} dx = 1 - \int_{x_1}^{\infty} e^{-x^2/2} dx = 1 - \Phi(-x_1), \tag{3.2.11}$$

which is used to simplify the valuation equation to

$$c = S_0 \left[1 - \Phi(\epsilon - \sigma\sqrt{T})\right] - Ke^{-rT}\left[1 - \Phi(\epsilon)\right] \tag{3.2.12}$$

The distribution can be rewritten by the property $\Phi(x_1) = 1 - \Phi(-x_1)$ as

$$1 - \Phi(\epsilon - \sigma\sqrt{T}) = \Phi(-\epsilon + \sigma\sqrt{T}) = \Phi\left\{\frac{1}{\sigma\sqrt{T}}\left[ln\left(\frac{S_0}{K}\right) - \left(r + \frac{\sigma^2}{2}\right)T\right]\right\} \tag{3.2.13}$$

This form is the standard Black-Scholes equation for a European Call option

$$c = S_o e^{-qT}\Phi(d_1) - e^{-rT}K\Phi(d_2), \tag{3.2.14}$$

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[ln\left(\frac{S_0}{K}\right) - \left(r - q + \frac{\sigma^2}{2}\right)T\right], \tag{3.2.15}$$

$$d_2 = \frac{1}{\sigma\sqrt{T}}\left[ln\left(\frac{S_0}{K}\right) - \left(r - q - \frac{\sigma^2}{2}\right)T\right] = d_1 - \sigma\sqrt{T}. \tag{3.2.16}$$

The Black-Scholes equation for a European Put option is

$$p = -S_o e^{-qT}\Phi(-d_1) + e^{-rT}K\Phi(-d_2), \tag{3.2.17}$$

with a dividend yield q. The same equations can be found from no-arbitrage arguments.

3.2.1.2. Black-Scholes Differential Equation

We start again from the SDE of the geometric Brownian motion

$$dS_t = \mu dt + \sigma dW_t \tag{3.2.18}$$

We will assume that the underlying randomness in the option price is the same source for the volatility in the stock price randomness [bemis2006]. The risk-free portfolio has a total value given by

$$\Pi_t = -f_{t,T} + \frac{\partial f_{t,T}}{\partial S} S_t \tag{3.2.19}$$

As changes in the stock price are linked to changes in the option contract price, the change in portfolio value at time t over a short time period is written as

$$\Delta\Pi_t = -\Delta f_{t,T} + \frac{\partial f_{t,T}}{\partial S} S_t \tag{3.2.20}$$

A perfectly hedged portfolio will earn the risk-free rate r

$$\Delta\Pi_t = r\Pi_t \Delta t = r(-f_{t,T} + \frac{\partial f_{t,T}}{\partial S} \partial S_t) \tag{3.2.21}$$

Ito's lemma [Appendix A.1] allows us to define the instantaneous price change of the option price that is a function of asset price and time as

$$df = \frac{\partial f}{\partial S} dS + \frac{\partial f}{\partial t} dt + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(dS)^2 \tag{3.2.22}$$

Substituting the spot price change model gives

$$df = \frac{\partial f}{\partial S}(\mu S_t dt + \sigma S_t dW_t) + \frac{\partial f}{\partial t} dt + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\left(\sigma^2 S_t^2 dW^2\right) \tag{3.2.23}$$

$$df = \left(\frac{\partial f}{\partial S}\mu S_t + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S_t^2\right) dt + \frac{\partial f}{\partial S}\sigma S_t dW \tag{3.2.24}$$

with the discrete version of the price change of the option price

$$\Delta f = \left(\frac{\partial f}{\partial S}\mu S_t + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S_t^2\right) \Delta t + \frac{\partial f}{\partial S}\sigma S_t \Delta W \tag{3.2.25}$$

The geometric Brownian process has a standard deviation of $1/\sqrt{years}$. This allows to be expressed $(dW)^2 = dt$ and $(\Delta W)^2 = \Delta t$.

$$\Delta\Pi_t = -\Delta f_{t,T} + \frac{\partial f_{t,T}}{\partial S} S_t = -\left\{\left(\frac{\partial f}{\partial S}\mu S_t + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S_t^2\right)\Delta t + \frac{\partial f}{\partial S}\sigma S_t \Delta W\right\}$$
$$+ \frac{\partial f}{\partial S}\frac{\partial f}{\partial S}(\mu S_t dt + \sigma S_t \Delta W_t \Longrightarrow \Delta\Pi_t = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S_t^2\right)\Delta t$$

Equating the risk-free growth of our portfolio to the change in value of our hedged portfolio

$$\Delta\Pi_t = r\Pi_t\Delta t \Longrightarrow \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S_t^2\right)\Delta t = r\left(-f + \frac{\partial f}{\partial S}S_t\right)\Delta t \tag{3.2.26}$$

gives the Black-Scholes differential equation (inserting a dividend rate q) for the evolution of the option price

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}(r-q)S_t + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S_t^2 = rf \tag{3.2.27}$$

3.2.1.3. Black-Scholes Option *Greeks* from Derived Formula

The term *Greeks* are the quantities representing the sensitivities of the price of derivatives such as options to a change in underlying parameters on which the value of an instrument or portfolio is dependent. Collectively these have also been called the risk sensitivities, risk measures or hedge parameters.

The Greeks in the Black–Scholes model are relatively easy to calculate and are very useful for derivatives traders, especially those who seek to hedge their portfolios from adverse changes in market conditions. The most common of the Greeks are the first order derivatives: *Delta*, *Vega* and *Theta* as well as *Gamma*, a second-order derivatives of the value function.

Delta *Delta* measures the rate of change of option value with respect to changes in the underlying asset's price. Delta is the first derivative of the value of the option with respect to the underlying instrument's price [Appendix A.2]

$$\Delta_{call} = \frac{\partial c}{\partial S} = e^{-qT}\Phi^{(n)}(d_1) \tag{3.2.28}$$

$$\Delta_{put} = \frac{\partial p}{\partial S} = e^{-qT}\Phi^{(n)}(-d_1) = e^{-qT}\left[\Phi^{(n)}(d_1) - 1\right] \tag{3.2.29}$$

Vega *Vega* measures sensitivity to volatility. Vega is the derivative of the option value with respect to the volatility of the underlying asset.

$$\nu_{call} = \frac{\partial c}{\partial \sigma} = S_0 e^{-qT}\phi^{(n)}(d_1)\sqrt{T} \tag{3.2.30}$$

$$\nu_{put} = \frac{\partial p}{\partial \sigma} = Ke^{-qT}\phi^{(n)}(d_2)\sqrt{T} \tag{3.2.31}$$

where ϕ is the probability density function (PDF).

Theta *Theta* measures the sensitivity of the value of the derivative to the passage of time or "time decay".

$$\theta_{call} = \frac{\partial c}{\partial t} = -S_0 e^{-rT} \frac{\phi^{(n)}(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}\Phi^{(n)}(d_2) + qS_0 e^{-qT}\Phi^{(n)}(d_1) \quad (3.2.32)$$

$$\theta_{put} = \frac{\partial p}{\partial t} = -S_0 e^{-rT} \frac{\phi^{(n)}(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}\Phi^{(n)}(-d_2) - qS_0 e^{-qT}\Phi^{(n)}(-d_1) \quad (3.2.33)$$

Gamma *Gamma* measures the rate of change in the delta with respect to changes in the underlying price. Gamma is the second derivative of the value function with respect to the underlying price.

$$\Gamma_{call} = \Gamma_{put} \rightarrow \frac{\partial^2 c}{\partial S^2} = \frac{\partial^2 p}{\partial S^2} = \frac{\partial \Delta}{\partial S} \rightarrow e^{-qT}\frac{\phi^{(n)}(d_1)}{S_0\sigma\sqrt{T}} \quad (3.2.34)$$

The $Black - Scholes$ differential equation for the evolution of the option price V is

$$\frac{\partial V}{\partial t} + rS_t\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} - rV = 0 \quad (3.2.35)$$

and can be rewritten with the option *Greeks*

$$\theta + rS_t\Delta + \frac{1}{2}\sigma^2 S_t^2\Gamma = rV \quad (3.2.36)$$

3.2.1.4. Hedging with Option *Greeks* in the Black-Scholes Framework

Hedging is the practice of taking a position in one market to offset and balance against the risk adopted by assuming a position in a contrary or other market or investment.

Option pricing and hedging theory are the core of modern mathematical finance since the derivation of the Black-Scholes formula, which provides a theoretical value and hedging strategy for European call/put options. Hedging with options is very common in energy applications simply because options are frequently the only instrument to address risk management needs. The key is that exists a trading strategy which constructs a portfolio that perfectly replicates the payoff of a call (or put) option under the following two assumptions

- The underlying risky asset price follows a geometric Brownian motion, and

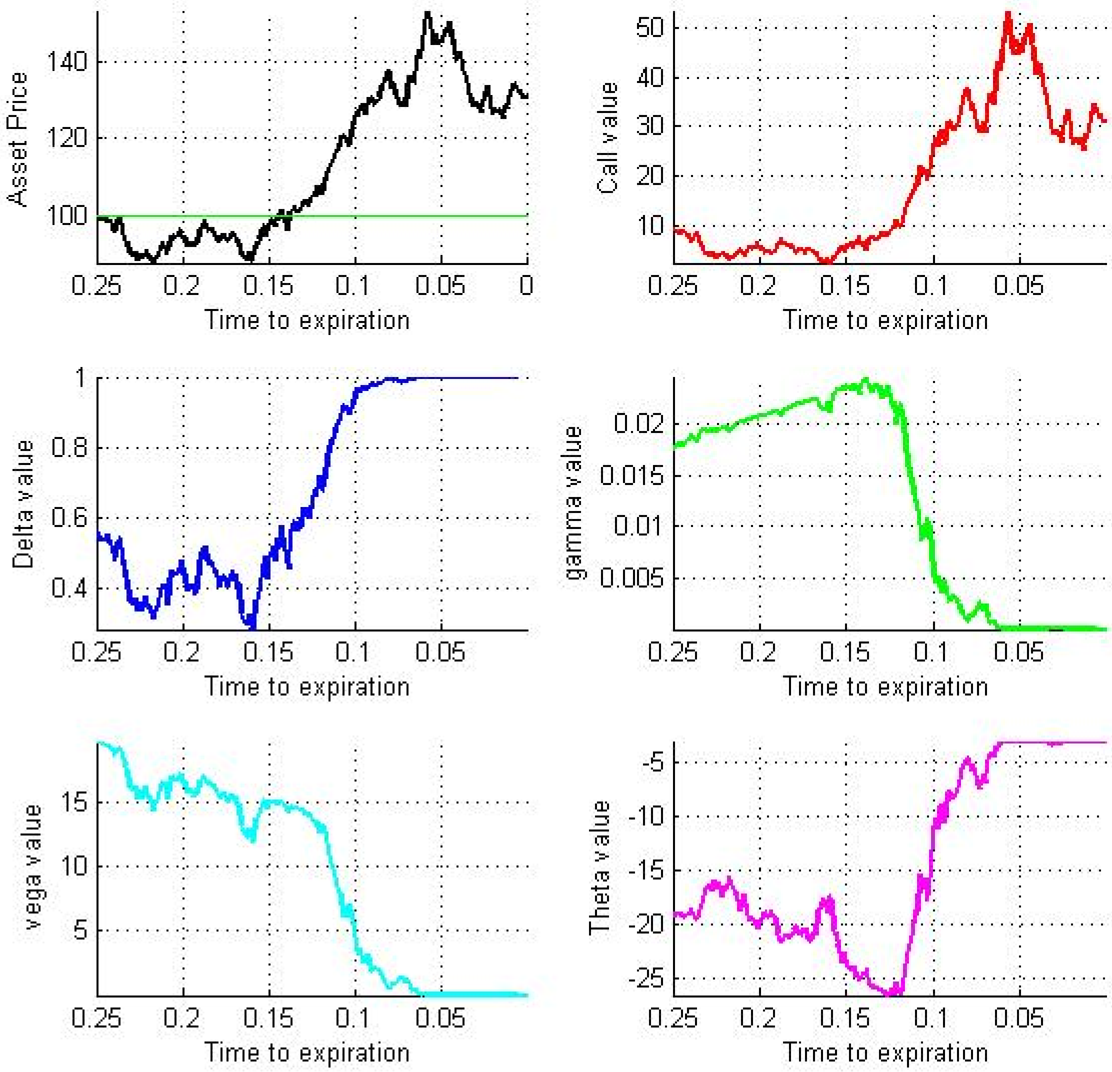


Figure 3.1.: Value of a call option depending to the asset price and time to expiration $\tau = T - t$. Monitoring of the standard Greeks depending also to τ.

- trading may take place in continuous time.

However, continues trading is never possible or even reasonable due to transactions costs (bid-ask spread, slippage, broker fees) and there always exist a hedging error i.e. perfect replication is not possible.

Dynamic Delta Hedging Delta neutral hedging is a strategy that aims to reduce (hedge) the risk associated with price movements in the underlying asset by offsetting long and short positions and set the value of $\Delta \backsimeq 0$ [3]. This strategy is based on the change in premium (price of option) caused by a change in the price of the underlying security. The change in premium for each basis-point change in price of the underlying is the delta and the relationship between the two movements is the hedge ratio.

For example, assume a portfolio which consists of a number of risk-free assets and a number of risky assets. Also, assume that we include a short[4] (sell) position in a European call option, C, with strike price K and exercise time T. Furthermore suppose that we have charged the arbitrage free price

$$\Pi_0[X] = e^{-rT} E^{\mathbb{Q}}[X] = C(S_0, K, \sigma, T) \tag{3.2.37}$$

where $\mathbb{Q}$ denotes the unique martingale measure (arbitrage-free) for the Black-Scholes model. In mathematical finance, an martingale measure is a risk-neutral measure and is heavily used in the pricing of financial derivatives due to the fundamental theorem of asset pricing, which implies that in a complete market a derivative's price is the discounted expected value of the future payoff under the unique risk-neutral measure [Holton2005].

In general when we short a call option we must be careful because the maximum loss is unlimited as the market rises and the maximum gain limited to the premium received for selling the option. We should use it when we are bearish[5] on market direction and also bearish on market volatility.

Now we would like to Delta-hedge this *short* position[6] in the call option, but due to restricted market access we can only rebalance our portfolio at the following discrete points in time: $t_0 = 0, t_1, ..., t_{n-1}$ where $t_j = jT/n$. This procedure is called as we described previously dynamic hedging. Suppose that our portfolio at time t_j consists of a number of risk-free assets and a number of risky assets. The value process for the hedge portfolio is then given by

$$V(t_j) = h_0(t_j)A(t_j) + h_1(t_j)S(t_j), \tag{3.2.38}$$

where $h_0(t_j)$ is the number of risk-free assets (e.g. amount of money invested in a bond or in a bank account with rate the risk-free rate) and $h_1(t_j)$ is the amount of units of the risky asset. Since we have also sold a call option, C, the value of our total portfolio is given by

[3] For example, a long call position may be delta hedged by shorting the underlying stock.

[4] A short is also known as a Naked Call. Naked calls are considered very risky positions because your risk is unlimited.

[5] Believing that a particular security, a sector, or the overall market is about to fall.

[6] Naked calls are tricky - if we take a substantial loss on a naked call option we will seriously evaluate our view on the stock. If we have a strong view on the stock making a bearish run we could buy put options to hedge our position or we could simply close out our naked call option by buying the same amount of call options.

$$\Pi(t_j) = V(t_j) - C(t_j, S(t_j)) = h_0(t)A(t_j) + h_1(t_j)S(t_j) - C(t_j, S(t_j)). \quad (3.2.39)$$

Here $C(t, S)$ denotes the pricing function for a European call option (i.e the premium). If we could rebalance[7] our portfolio continuously we would obtain a perfect hedge if we always had $h_1(t) = \Delta_X(t) = \frac{\partial C}{\partial S}$ stocks in our portfolio. This cannot be done due to market restrictions. Requiring the total portfolio to be Delta neutral gives the following equation to solve for $h_1(t_j)$

$$\Delta_\Pi(t_j) = \frac{\partial \Pi}{\partial S} = \frac{\partial}{\partial S}(h_0(t)A(t_j)+h_1(t_j)S-C(t_j, S)) = h_1(t_j)-\Delta_C(t_j) = 0. \quad (3.2.40)$$

Using this strategy we will not be able to replicate the contract perfectly, and therefore there will be a **hedging error**

As the price of the underlying asset drops, the Delta of the call follows suit. We are therefore selling our holdings gradually, recovering some funds for our bank account balance. Eventually the price recovers and we build up asset holding once more. The Delta value will mimic the process of the call option to a large extent, but not exactly.

Now we are interested in the distribution of the hedging error, i.e. the distribution of

$$\Pi_T = V(T) - C, \quad (3.2.41)$$

where Π_T is the value of the total portfolio at time T , $V(T)$ is the value of the delta hedge portfolio at time T , and C is the value of the call option at time T .

Increasing the frequency of trades will decrease the volatility of this hedging error, and of course at the limit case the replicating strategy is exact. If from one transaction to the next the Delta does not move a lot, we would expect the impact of discrete hedging to be small. On the other hand, the impact will be most severe in the areas where the Delta itself changes rapidly. These effects summarized to the second order sensitivity with respect to the price, the Gamma.

Value at Risk (VaR) In risk management, it is of great importance to have a coherent and widely recognized measure to assess financial market risk. The Value at Risk (VaR) framework intends to fulfill this need. It was first introduced in 1989

[7] There is one very strong argument against becoming a rebalancing addict, and that is cost. We assumed all distributions were reinvested along the way and did not factor in transaction fees or taxes on realized gains from trades. But these are both very important issues to consider when thinking about a rebalancing strategy.

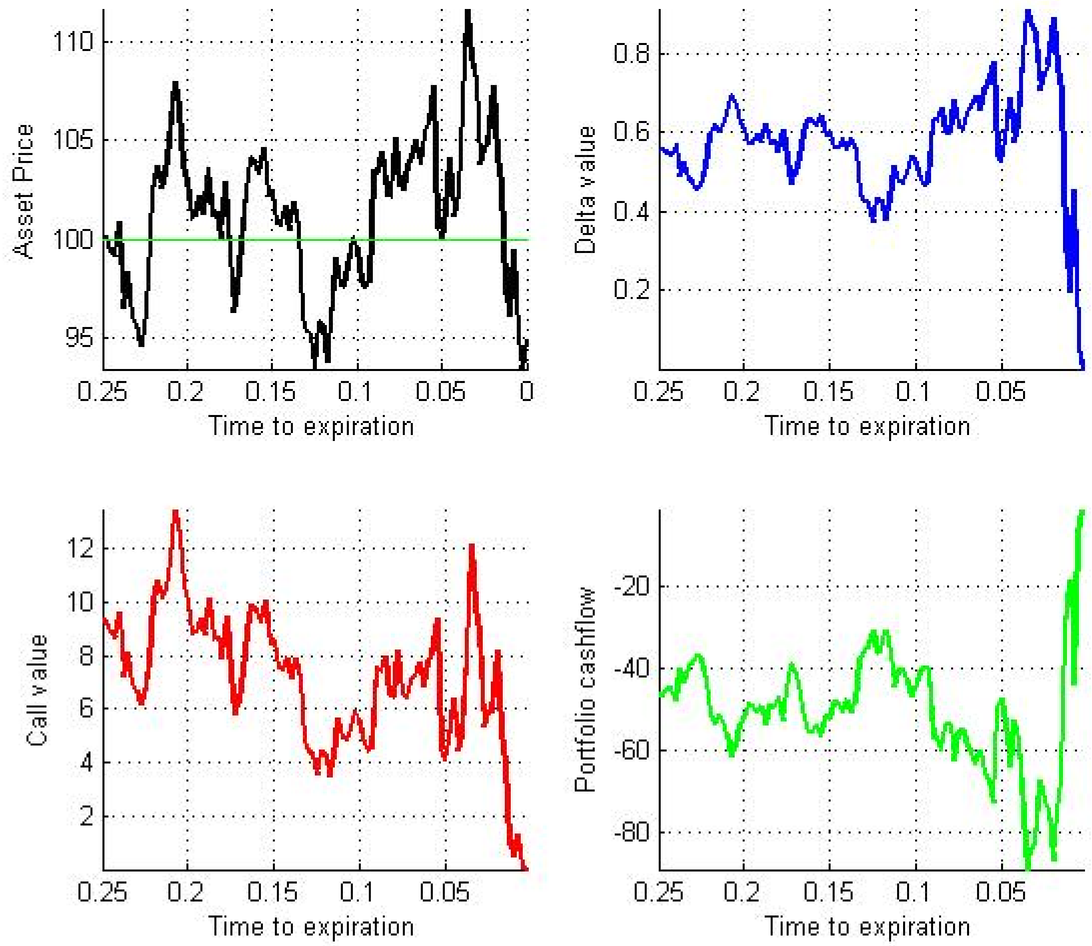


Figure 3.2.: Dynamic Delta hedging in a simulated Black-Scholes enviroment. We can see the evaluation of the portfolio's delta and our cashflow (e.g. bank account) depenting to τ.

by J.P. Morgan to have a concise view of the risks of the firm. The aim of the VaR is to have a single figure that gives an immediate and understandable overview of the risk of a certain financial instrument. In the 90s, the VaR quickly established itself as a benchmark in the financial industry, [Jorion2001], [McNeil2005].

The Value at Risk (VaR) is defined as the α quantile of a return distribution and gives an indication of the loss threshold at an α-level one should expect when investing in a certain instrument or portfolio over a pre-defined time horizon. Typically, $1-\alpha$ is chosen to be either $1-\alpha = 95\%$ or $1-\alpha = 99\%$. Hence, in the case of the $VaR_{95\%}$, $1-\alpha = 95\%$ of the returns for the considered period should be above the value and $\alpha = 5\%$ below.

When computing the daily VaR, one needs to forecast the return distribution for the next day. Several approaches exist. Among them it is worthwhile to cite the historical method, GARCH models [Engle1982], Monte-Carlo simulation [Glasserman2000] or multi-fractals [Boisbourdain2008].

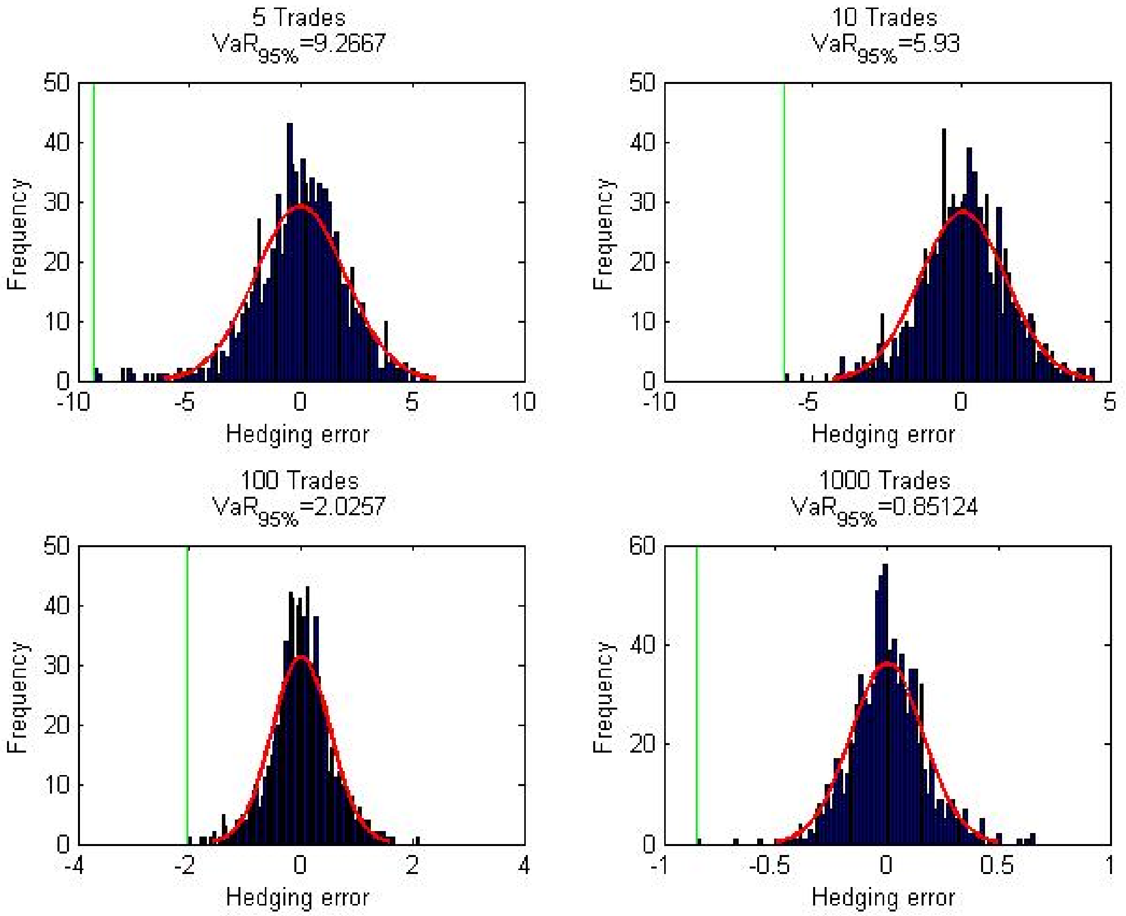


Figure 3.3.: Monte-Carlo simulation of Delta hedging error and $VaR_{95\%}$ (green line). Number of simulated paths $N = 1000$ and $K = 100$, $T = 1$, $r = 0.03$ and $\sigma = 0.3$. In the above figure, we can observe that as the rebalances of our portfolio are increased the delta hedging error decreases. In an ideal world that the rebalances $\to \infty$, the hedging error $\to 0$.

Gamma (Neutral) Hedging The Γ(Gamma) describes how sensitive is Δ to small changes in S. To obtain an even smaller hedging error, one can use a Gamma hedge strategy. This makes the portfolio both Delta and Gamma neutral, i.e.

$$\Gamma_\Pi = \frac{\partial^2 \Pi}{\partial S^2} = 0, \qquad \Delta_\Pi = \frac{\partial \Pi}{\partial S} = 0. \tag{3.2.42}$$

This can be done by allowing the portfolio to contain one more asset. Assume that this asset is a European call, C_2, like the one we're trying to hedge (C_1), only this one has a different time to maturity. So now we introduce a third portfolio process, $h_2(t_j)$, that symbolizes the number of shares held in this contract. The conditions on Γ_Π and Δ_Π impose the following conditions on the portfolio:

$$\begin{aligned} h_2(t_j) &= \Gamma_{C_1}(t_j)/\Gamma_{C_2}(t_j) \\ h_1(t_j) &= \Delta_{C_1}(t_j) - h_2(t_j)\Delta_{C_2}(t_j) \end{aligned} \tag{3.2.43}$$

So now we have the following value process

$$\begin{aligned} h_0(t_j)A(t_j) + h_1(t_j)S(t_j) + h_2(t_j)V(t_j, C_2) = \\ = h_0(t_{j+1})A(t_j) + h_1(t_{j+1})S(t_j) + h_2(t_{j+1})V(t_j, C_2) \end{aligned} \tag{3.2.44}$$

where $V(t_j, C_2)$ denotes the time t_j value of the claim C_2.

Vega (volatility) hedging Vega (volatility)[8] hedging adjust the volatility exposure. Under the assumption of the Black-Scholes model, Vega hedging is not necessary because σ does not change. But this assumption is resolutely invalid in the case of energy derivatives. Consequently, to create realistic pricing models in energy markets, it is essential to understand the behavior of implied volatility.

Application: Delta and vega hedging Take a Black-Scholes' delta hedging strategy for a call option:

$$\Delta = \frac{\partial C_{BS}(S, K, \tau, \sigma)}{\partial S} = \left\{ \begin{array}{ll} \frac{\partial C_{BS}}{\partial S} & \text{if } \sigma \; is \; constant \\ \frac{\partial C_{BS}}{\partial S} + \frac{\partial C}{\partial \sigma} \underbrace{\frac{\partial \sigma}{\partial S}}_{vega} & \text{if } \sigma \text{ varies} \end{array} \right\} \tag{3.2.45}$$

Consider an option portfolio that is delta-neutral but with a vega of -8.000. We plan to make the portfolio both delta and vega neutral using two instruments: The underlying stock and a traded option with delta 0.6 and vega 2.0.

To achieve a vega neutral simulation, we need long $8000/2 = 4.000$ contracts of the traded option. With the traded option added to the portfolio, the delta of the

[8] implied volatility of an option contract is that value of the volatility of the underlying instrument which, when input in an option pricing model (such as Black-Scholes) will return a theoretical value equal to the current market price of the option.

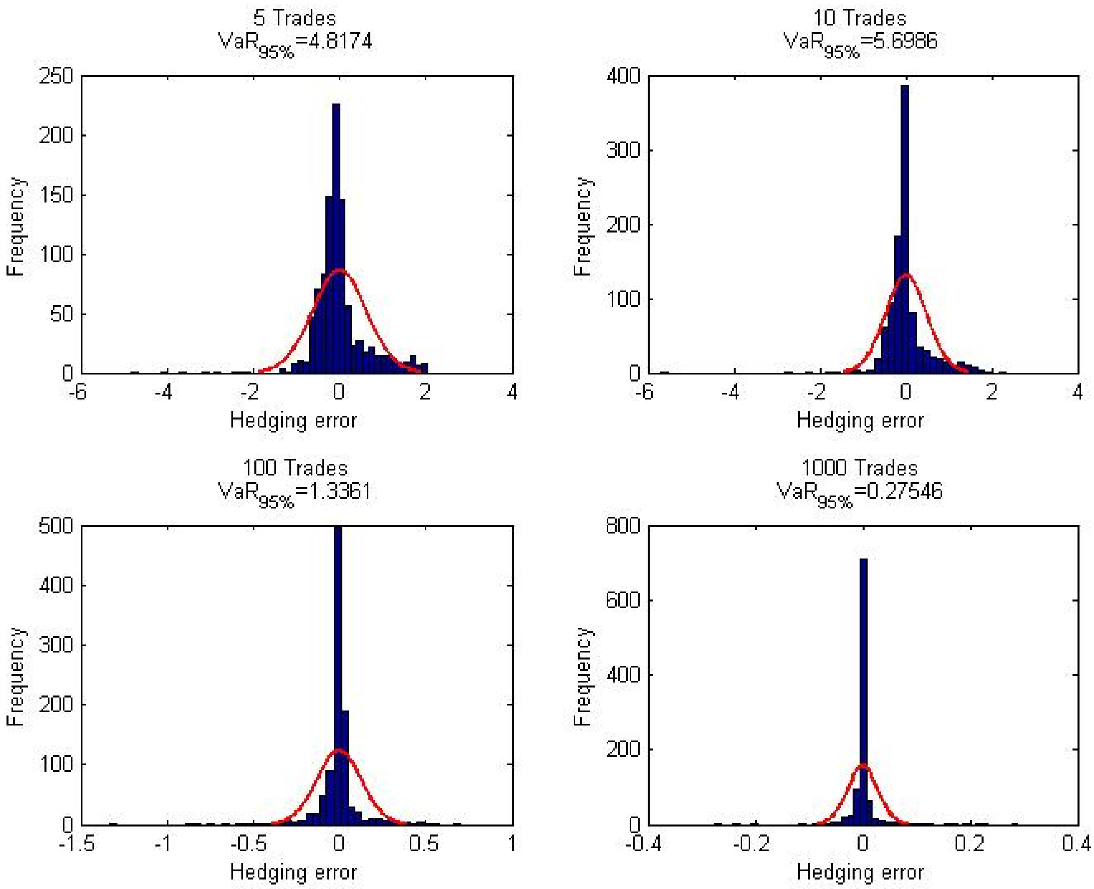


Figure 3.4.: Distributions of the Delta-Gamma hedging error. Parameter set similar to Delta hedging with $T_1 = 1$ and $T_2 = 1.1$

portfolio increases from 0 to $0.6 \times 4.000 = 2.400$. We hence also need to short 2.400 shares of the underlying stock $\Rightarrow$ each share of the stock has a delta of one.

By controlling the vega risk, the hedge error can be reduced by 30%-40% [Derman2006]

3.2.1.5. Hedging In Practice

- Traders usually ensure that their portfolios are Delta-neutral at least once a day. For this reason Delta (neutral) hedging is very important.
- As portfolio becomes larger, hedging becomes less expensive.
- Profits from Black-Scholes PDE: $\theta + rS_t\Delta + \frac{1}{2}\sigma^2 S_t^2\Gamma = rV$

3.2.2. Heston's Stochastic Volatility Model

The use of stochastic volatility models to evaluate prices of financial derivatives among market practitioners has increased in the past few years. These models provide a better calibration to market-implied volatility smiles and skews whilst providing realistic dynamics to the underlying stock. The Heston model has become particularly popular because of the availability of closed-form formulas for the Fourier transform of the price of European options. Unlike the Black-Scholes model where the instantaneous volatility process of the asset prices is assumed to be deterministic, the Heston model describes the volatility process using a mean-reverting square root process.

3.2.2.1. Derivation of the Heston's Model

Heston (1993) implemented a stochastic square root of variance $\sqrt{u}$, factor in the SDE for the evolution of the asset price as

$$dS_t = \mu S_t dt + \sqrt{u_t} S_t dW_1. \tag{3.2.46}$$

In the Heston model, the square root of variance is assumed to follow an Ornstein-Uhlenbeck process

$$d\sqrt{u_t} = -\beta\sqrt{u_t}dt + \delta dW_2, \tag{3.2.47}$$

where β is the reversion rate (to zero) and δ is the volatility of the square-root process. Transforming the square-root stochastic equation into a stochastic equation of a linear function via the general function $G(x) = x^2$, which has the following derivatives:

$$\frac{\partial G}{\partial x} = 2x, \qquad \frac{\partial^2 G}{\partial x^2} = 2, \qquad \frac{\partial G}{\partial t} = 0. \tag{3.2.48}$$

Application of Ito's lemma [see Appendix A.1] gives us

$$dG = \frac{\partial G}{\partial x}dx + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}(dx)^2 + \frac{\partial G}{\partial t}dt + \frac{1}{2}\frac{\partial^2 G}{\partial t^2}\overbrace{(dt)^2}^{dt\to 0} \tag{3.2.49}$$

$$dG = 2xdx + (dx)^2 \tag{3.2.50}$$

Now assigning $x = \sqrt{u_t}$ gives $G = u_t$and

$$du_t = 2\sqrt{u_t}\left\{-\beta\sqrt{u_t}dt + \delta dW_2\right\} + \left(-\beta\sqrt{u_t}dt + \delta dW_2\right)^2. \tag{3.2.51}$$

Expanding and removing the insignificant terms gives

$$du_t = \{-2\beta u_t dt + 2\delta\sqrt{u_t}dW_2\} + \delta^2 \underbrace{(dW_2)^2}_{:=dt}, \tag{3.2.52}$$

$$du_t = \left\{\delta^2 - 2\beta u_t\right\} dt + 2\delta\sqrt{u_t}dW_2. \tag{3.2.53}$$

Now assigning $k = 2\beta$ as the volatility reversion rate, $\theta = \frac{\delta^2}{2\beta}$ as the long-term variance level, and $\xi = 2\delta$ as the volatility of the volatility we have

$$du_t = k(\theta - u_t)dt + \xi\sqrt{u_t}dW_2. \tag{3.2.54}$$

So we can summarize the Heston dynamic model by three equations

$$dS_t = \mu S_t dt + \sqrt{u_t}S_t dW_1, \tag{3.2.55}$$

$$du_t = k(\theta - u_t)dt + \xi\sqrt{u_t}dW_2, \tag{3.2.56}$$

$$E\left[dW_1 dW_2\right] = \rho dt, \tag{3.2.57}$$

where μ is the rate of return of the asset, θ is the mean-reversion or long-term variance level (as $t \to \infty$ the expected value $u_t \to \theta$), k is the volatility reversion rate at which u_t reverts to θ, ξ is the volatility of volatility. The two Wiener processes are correlated by a constant factor ρ, and u_0 is the initial volatility. The Heston model for a constant volatility reduces to the Black-Scholes model.

3.2.2.2. Heston Partial Differential Equation

The Black-Scholes model assumes a single source of randomness. Inspection of the Heston model shows two sources of randomness. Thus, it is necessary to theoretically hedge the risk sources with two options. The risk-free portfolio has a total value given by

$$\Pi_t = V_t + \Delta S_t + \Xi U_t, \tag{3.2.58}$$

where V is the value of an option, Ξ shares of another option, and Δshares of asset stock. The change in portfolio's value at time t over a short time period is given by

$$d\Pi_t = dV_t + \Delta dS_t + \Xi dU_t. \tag{3.2.59}$$

Now apply Ito's lemma for the option V

$$
\begin{aligned}
dV &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS_t + \frac{\partial V}{\partial u}du_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS_t)^2 + \\
&\quad \frac{1}{2}\frac{\partial^2 V}{\partial u^2}(du_t)^2 + \frac{\partial^2 V}{\partial u \partial S}du_t dS_t + \frac{\partial^2 V}{\partial u \partial S}du_t dS_t
\end{aligned}
$$

Substituting the Heston model dynamics and with the knowledge that $(dW_1)^2 = (dW_2)^2 = dt$ and $dW_1 dW_2 = \rho dt$ gives

$$
\begin{aligned}
dV &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial u}du + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}uS^2 dt + \\
&\quad \frac{1}{2}\frac{\partial^2 V}{\partial u^2}\sigma^2 u + \sigma u \rho S \frac{\partial^2 V}{\partial u \partial S}dt
\end{aligned}
$$

Inserting the expression for dV (identical for dU) into the change of portfolio equation gives

$$
\begin{aligned}
d\Pi_t &= \left\{\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}uS^2 + \frac{1}{2}\frac{\partial^2 V}{\partial u^2}\sigma^2 u + \sigma u \rho S \frac{\partial^2 V}{\partial u \partial S}\right\} dt + \\
&\quad \Xi \left\{\frac{\partial U}{\partial t} + \frac{1}{2}\frac{\partial^2 U}{\partial S^2}uS^2 + \frac{1}{2}\frac{\partial^2 U}{\partial u^2}\sigma^2 u + \sigma u \rho S \frac{\partial^2 U}{\partial u \partial S}\right\} dt \\
&\quad + \underbrace{\left\{\frac{\partial V}{\partial S} + \Xi \frac{\partial U}{\partial S} + \Delta\right\} dS + \left\{\frac{\partial V}{\partial u} + \Xi \frac{\partial U}{\partial u}\right\} du}_{:=0 \rightarrow To\, form\, a\, riskless\, portofolio}
\end{aligned}
$$

A perfectly hedged portfolio must earn the risk-free rate r and given by

$$
d\Pi_t = r\Pi_t \Delta t = r(V + \Delta S + \Xi U)dt. \tag{3.2.60}
$$

Equating the previous two equations and substituting the hedged share ratios given

$$
\begin{aligned}
&\frac{\left\{\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}uS^2 + \frac{1}{2}\frac{\partial^2 V}{\partial u^2}\sigma^2 u + \sigma u \rho S \frac{\partial^2 V}{\partial u \partial S}\right\} - rV + rS\frac{\partial V}{\partial S}}{\frac{\partial V}{\partial u}} = \\
&= \frac{\left\{\frac{\partial U}{\partial t} + \frac{1}{2}\frac{\partial^2 U}{\partial S^2}uS^2 + \frac{1}{2}\frac{\partial^2 U}{\partial u^2}\sigma^2 u + \sigma u \rho S \frac{\partial^2 U}{\partial u \partial S}\right\} - rU + rS\frac{\partial U}{\partial S}}{\frac{\partial U}{\partial u}}
\end{aligned}
$$

where the left-hand side contains terms of V and the right-hand side contains terms of U. The right hand side is equal to the left hand side where both sides are equal to a function $f(S, u, t)$ with independent variables S, u, t. Heston (1993) gives this function as

$$f(S, u, t) = -k(\theta - u_t) + \lambda(S, u, t), \tag{3.2.61}$$

where $\lambda(S, u, t)$ is the price of volatility risk. Equating the function $f(S, u, t)$ to the "V" side and rearranging gives the $\mathbf{Heston\, linear - price\ PDE}$ as

$$\left\{\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}uS^2 + \frac{1}{2}\frac{\partial^2 V}{\partial u^2}\sigma^2 u + \sigma u \rho S \frac{\partial^2 V}{\partial u \partial S}\right\}$$
$$-rV + rS\frac{\partial V}{\partial S} + \left\{k(\theta - u_t) - \lambda(S, u, t)\right\}\frac{\partial V}{\partial u} = 0,$$

for $0 \le t \le T$, $S > 0$, $V > 0$. The Heston PDE can be viewed as a time dependent convection-diffusion-reaction equation, on an unbounded two-dimensional spatial domain. The parameter $\kappa > 0$ is the mean-reversion rate, $\theta > 0$ is the longterm mean, $\sigma > 0$ is the volatility-of-volatility, $\rho \sim [-1, 1]$ is the correlation between the two underlying Brownian motions, and r is the interest rate. In this paper we always assume that $2\kappa\theta > \sigma^2$, which is known as the Feller condition.

3.2.2.3. Decoupled Green Function Approach to the Heston Model

The Heston PDE for the value of a derivative on underlying asset S was derived as

$$\left\{\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}uS^2 + \frac{1}{2}\frac{\partial^2 V}{\partial u^2}\sigma^2 u + \sigma u \rho S \frac{\partial^2 V}{\partial u \partial S}\right\}$$
$$-rV + rS\frac{\partial V}{\partial S} + \underbrace{\left\{k(\theta - u_t) - \lambda(S, u, t)\right\}}_{b(u_t)}\frac{\partial V}{\partial u} = 0,$$

for $0 \le t \le T$, $S > 0$, $V > 0$. For convenience, define $a(u_t) = \sigma\sqrt{u_t}$ and $b(u_t) = k(\theta - u_t) - \lambda(S, u, t)$. The transform equations

$$\tau = T - t \tag{3.2.62}$$

$$x = ln(S) + r\tau = ln(Se^{r\tau}) = ln(F_T) \tag{3.2.63}$$

$$V = W(x, u, \tau)e^{-rt} \tag{3.2.64}$$

produce the PDE given by

$$\frac{1}{2}u\left(\frac{\partial^2 W}{\partial x^2}-\frac{\partial W}{\partial x}\right)+\rho\sqrt{u}a(u)\frac{\partial^2 W}{\partial x\partial u}$$
$$+\frac{1}{2}a^2(u)\frac{\partial^2 W}{\partial u^2}+b(u)\frac{\partial W}{\partial u}=\frac{\partial W}{\partial \tau},$$

if we make use of the Fourier transform and inverse Fourier transform pair to analyze the Heston PDE in Fourier space we have:

$$W(x;u,\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\omega x\widetilde{W}(\omega;u,\tau)}, \tag{3.2.65}$$

$$\tilde{W}(x;u,\tau)=\int_{-\infty}^{\infty}e^{i\omega x}W(x;u,\tau)dx. \tag{3.2.66}$$

Applying $\frac{\partial \tilde{W}}{\partial x}=-i\omega\tilde{W}$ to the PDE for W gives

$$-\frac{1}{2}u(\omega^2\tilde{W}-i\omega\tilde{W})+[-i\omega\rho\sqrt{u}a(u)+b(u)]\frac{\partial\tilde{W}}{\partial u}$$
$$+\frac{1}{2}a^2(u)\frac{\partial^2\tilde{W}}{\partial u^2}=\frac{\partial\tilde{W}}{\partial\tau}.$$

3.2.2.4. Fourier Terminal Payoff & Heston's Fundamental Solution (Green function)

In Hestons's stochastic volatility framework, the main problem when implementing Heston's semi-analytic formulas for European style options is the inverse Fourier integration (Handbook of Quantitative Finance, Cheng F. Lee, Alice C). We consider the construction of the Fundamental Solution of Heston's model in Fourier space. The terminal condition of the Fourier transform is

$$\tilde{W}(\omega;\tau=0)=\int_{-\infty}^{\infty}e^{i\omega x}W(x;u,\tau=0)dx=\int_{-\infty}^{\infty}e^{i\omega x}V(x,u,\tau=0)dx. \tag{3.2.67}$$

The payoff of a European vanilla call option is $V(x,u,\tau=0)=(e^x-K)^+$, thus

$$\tilde{W}(\omega;\tau=0)=\int_{-\infty}^{\infty}e^{i\omega x}(e^x-K)^+dx$$
$$=\int_{ln(K)}^{\infty}e^{(1+i\omega)x}-Ke^{i\omega x}=\left[\frac{e^{(1+i\omega)x}}{1+i\omega}-K\frac{e^{i\omega x}}{i\omega}\right]_{ln(K)}^{\infty}.$$

The solution V of the option is composed of the Green function, $\tilde{G}$, and a (volatility independent) time T payoff function. Both components defined in Fourier space as

$$V(x;u,\tau) = \frac{1}{2\pi}e^{-r\tau}\int_{iIm(\omega)-\infty}^{iIm(\omega)+\infty} e^{-i\omega x}\tilde{W}(\omega;\tau=0)\tilde{G}(\omega;u,\tau)d\omega, \qquad (3.2.68)$$

where the terminal condition $\tilde{G}(\omega;u,\tau=0)=1$. The function G is the **fundamental tranform** (or Green function) and is assumed to have an affine form of the type

$$\tilde{G}(\omega;u,\tau) = e^{C(\tau,\omega)+uD(\tau,\omega)}. \qquad (3.2.69)$$

G is a solution to the PDE (similar derived for $\tilde{W}$)

$$\begin{aligned}\frac{\partial\tilde{G}}{\partial\tau} &= \frac{1}{2}a^2(u)\frac{\partial^2\tilde{G}}{\partial u^2} - \frac{1}{2}u(\omega^2-i\omega)\tilde{G} + (-i\omega\rho\sqrt{u}a(u)+b(u))\frac{\partial\tilde{G}}{\partial u}\\ &= \frac{1}{2}\sigma^2 u\frac{\partial^2 G}{\partial u^2} - \frac{1}{2}u(\omega^2-i\omega)\tilde{G} + [k(\theta-u)-\lambda u - i\omega\rho\sigma u]\frac{\partial\tilde{G}}{\partial u}.\end{aligned}$$

Since the terminal condition $\tilde{G}(\omega;u,\tau=0)=1$ is known and $C(\tau=0,u)=0$ and $D(\tau=0,u)=0$ for consistency. Replacing the partial derivatives of $\tilde{G}$ in the previous PDE gives

$$\left\{\frac{\partial C}{\partial\tau} + u\frac{\partial D}{\partial\tau}\right\} = \frac{1}{2}\sigma^2 uD^2 - \frac{1}{2}u(\omega^2-i\omega) + [k(\theta-u)-\lambda u - i\omega\rho\sigma u]D. \qquad (3.2.70)$$

In this differential equation, the terms independent of u are

$$\frac{\partial C}{\partial\tau} = \kappa\theta D, \qquad (3.2.71)$$

and dependent on u are

$$\frac{\partial D}{\partial\tau} = \frac{1}{2}\sigma^2 D^2 - \frac{1}{2}u(\omega^2-i\omega) + [k+\lambda+i\omega\rho\sigma]D. \qquad (3.2.72)$$

The final expressions for C and$D$ of Green's function are

$$D = \frac{\kappa+\lambda+i\omega\rho\sigma}{\sigma^2}\left(\frac{1-e^{d\tau}}{1-ge^{d\tau}}\right),$$

where

$$d = \sqrt{\sigma^2(\omega^2 - i\omega) + [k + \lambda + i\omega\rho\sigma]^2}, \tag{3.2.73}$$

and

$$C = \frac{\kappa\theta}{\sigma^2}\left[(k + \lambda + i\omega\rho\sigma + d)\,\tau - 2ln\left(\frac{1 - ge^{d\tau}}{1 - g}\right)\right],$$

where

$$g = \frac{k + \lambda + i\omega\rho\sigma + d}{k + \lambda + i\omega\rho\sigma - d}. \tag{3.2.74}$$

We employ an implementation of Heston's semi-analytical formula to acquire values of V in a 40x40 grid. For calculating the single integrals occuring in (3.2.68) in MATLAB® version R2008b we use a numerical quadrature rule **quadl** and in the newest version R2013a we use the function **integral** which minimize our computation time.

3.2.2.5. Heston Greeks

Delta is the first derivative with respect to underlying asset price S

$$\Delta = \frac{\partial V(x; u, \tau)}{\partial S} = \frac{-i\omega}{S} V(x; u, \tau) \tag{3.2.75}$$

Similarly, Gamma is the second derivative with respect to underlying asset price S

$$\Gamma = \frac{\partial^2 V(x; u, \tau)}{\partial S^2} = \frac{-\omega^2}{S^2} V(x; u, \tau) \tag{3.2.76}$$

Vega is the first derivative with respect to underlying asset price variance u

$$Vega = \frac{\partial V(x; u, \tau)}{\partial u} = \frac{1}{2\pi} e^{-r\tau} \int_{iIm(\omega)-\infty}^{iIm(\omega)+\infty} e^{-i\omega x} \tilde{W}(\omega; \tau = 0) \frac{\partial}{\partial u} \tilde{G}(\omega; u, \tau) d\omega \tag{3.2.77}$$

with

$$\frac{\partial}{\partial u}\tilde{G}(\omega; u, \tau) = \frac{\partial}{\partial u}(e^{C(\tau,\omega) + uD(\tau,\omega)}) = D(\tau, \omega)\tilde{G}(\omega; u, \tau) \tag{3.2.78}$$

Vega is given by

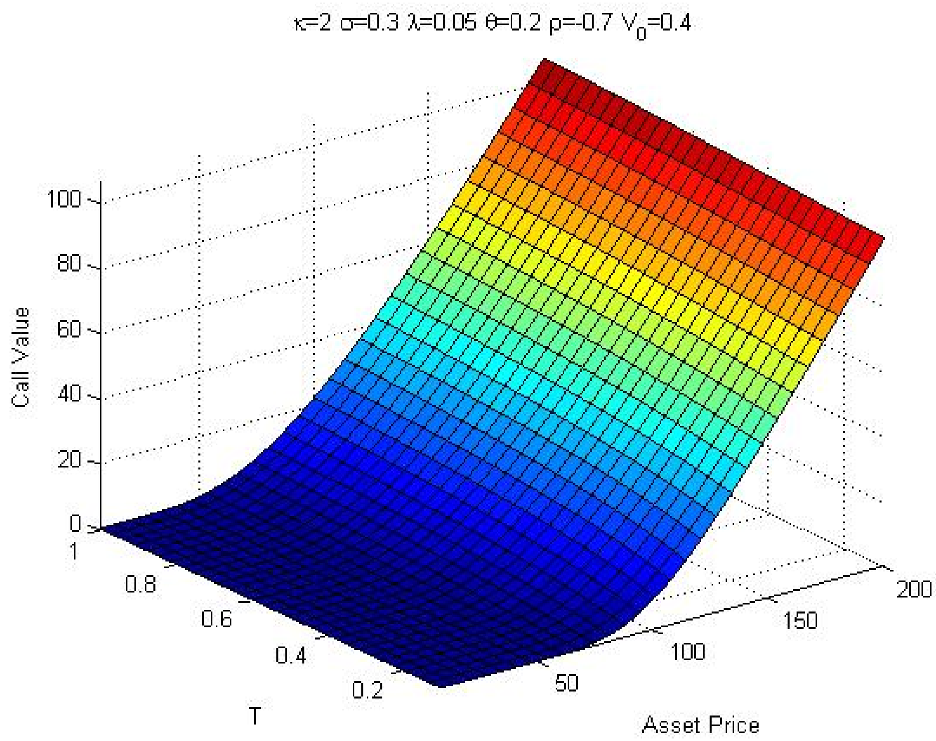


Figure 3.5.: European call price as a function of time to expiration and initial asset price calculated using a Fourier inversion of the fundamental transform Heston model

$$Vega = \frac{\partial V(x;u,\tau)}{\partial u}$$
$$= \frac{1}{2\pi} e^{-r\tau} \int_{iIm(\omega)-\infty}^{iIm(\omega)+\infty} e^{-i\omega x} \tilde{W}(\omega;\tau=0) \left\{ D(\tau,\omega) \frac{\partial}{\partial u} \tilde{G}(\omega;u,\tau) \right\} d\omega$$

An advantage of the Heston model is its ability to replicate the volatility skew present in market option data as implied from the Black-Scholes equation. This skew arises from the negative correlation, typically $-1 < u < -0.7$, between the volatility to the asset price.

As discussed, the Black-Scholes model ignores two possible behaviors of the stock process, and those are discontinuity of the stock process and changes in volatility. One way to handle the second one and to model our market in a more realistic way is to use stochastic volatility models such as the Heston model. However the market

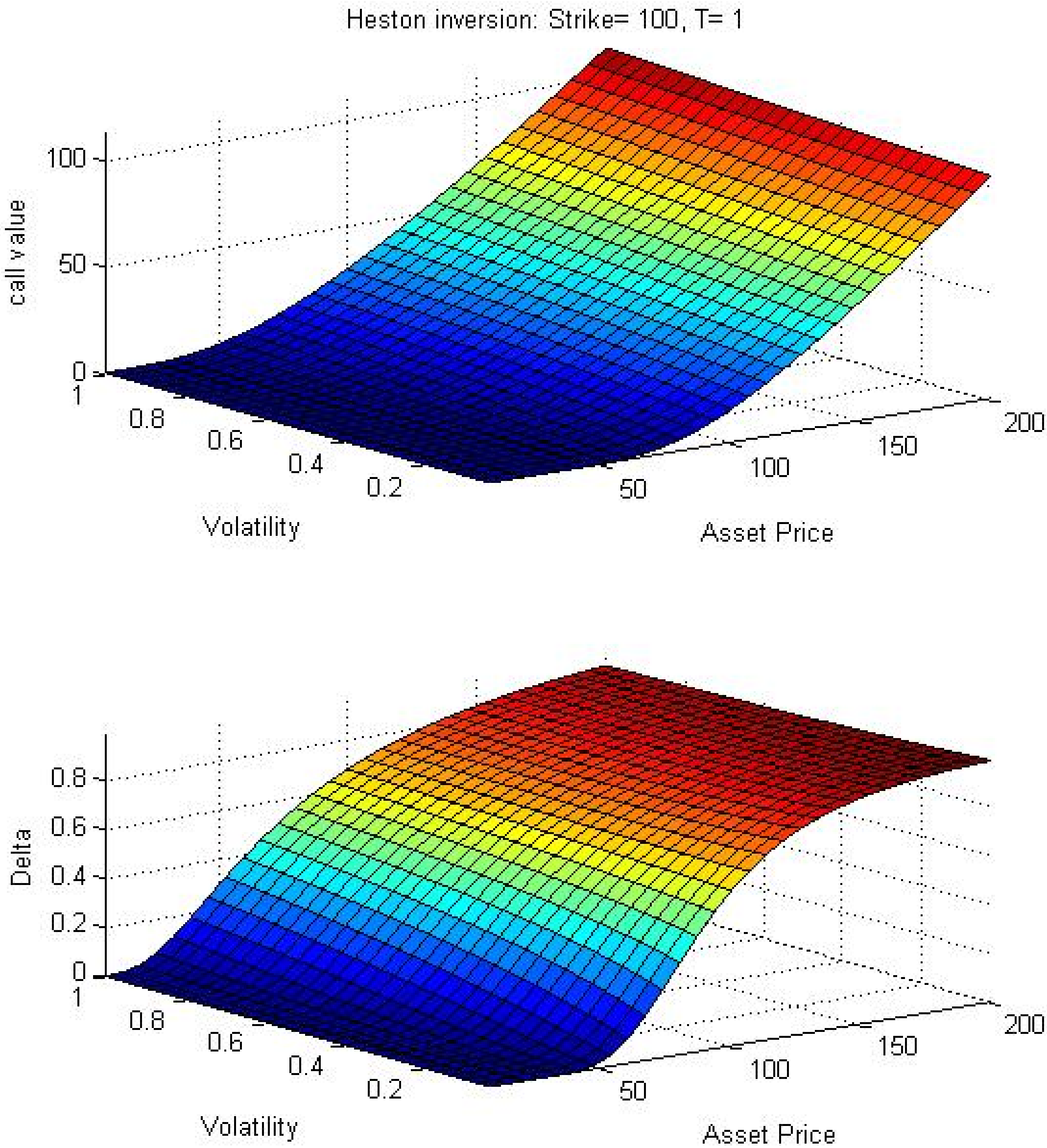
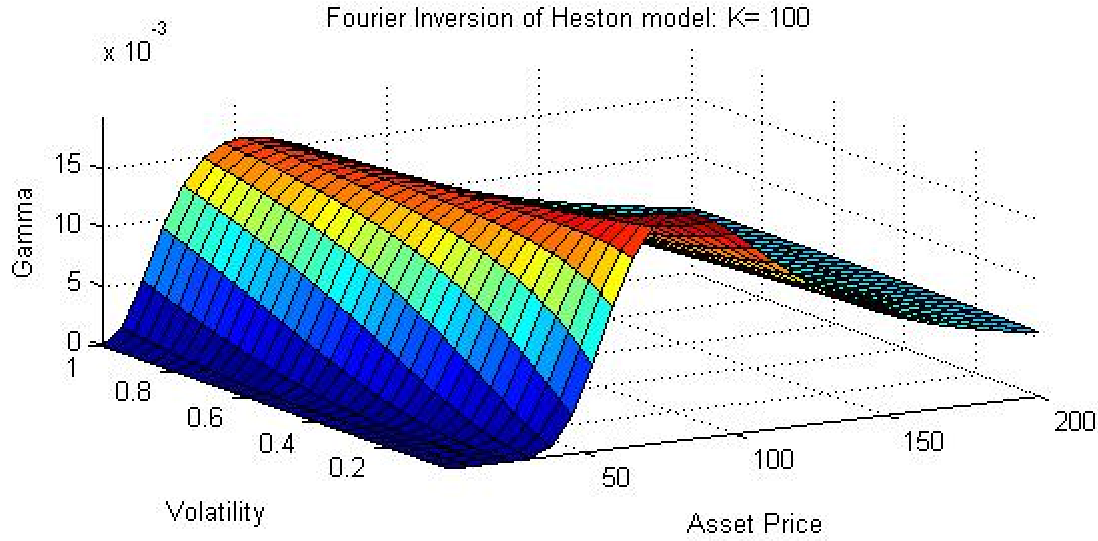


Figure 3.6.: (Top) Call price and option (bottom) Delta as a function of initial volatility (v_0 =0.2) and initial asset price calculated via Fourier inversion of the Heston model

still uses the Black-Scholes formula in order to price traded derivatives. The question is which value of volatility we should include in to the Black-Scholes formula in order to obtain the right option price.

After solving the Heston PDE we have calculated the option price in terms of an



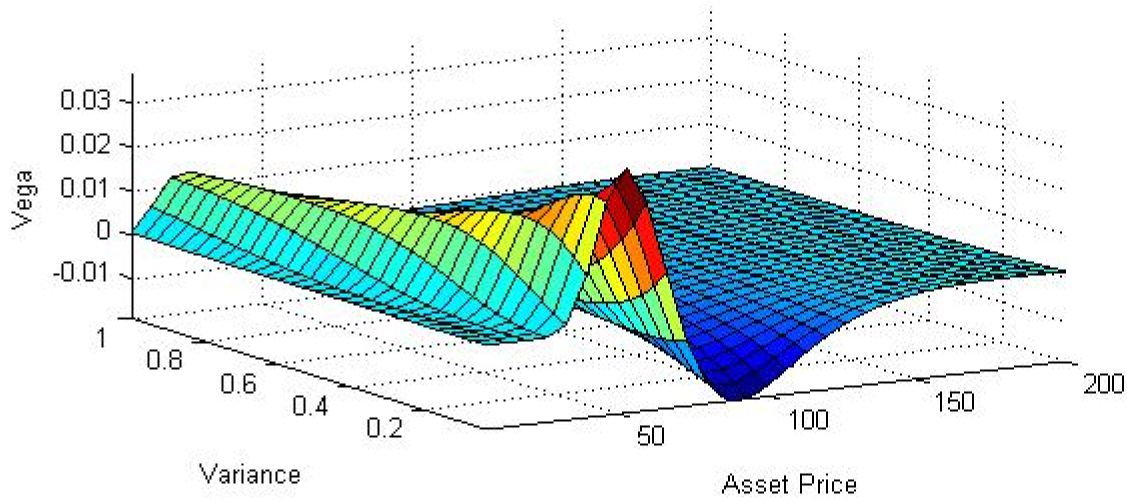


Figure 3.7.: (Top) Gamma and (bottom) Vega as a function of initial variance (v_0=0.3) and initial asset price calculated via Fourier inversion of the Heston model.

underlying asset. This asset has some volatility that varies over time. Therefore all we have to do to answer the question above is to equalize the price of the derivative we have calculated with the Black-Scholes formula and solve for the volatility. The solution to this equation will give us the implied volatility that corresponds to the option price that we calculated. In this sense we can say that the importance of the implied volatility is that it is that value of volatility which gives us a more accurate price of the derivative we are pricing.

The implied volatility cannot be calculated analytically because the Black-Scholes model cannot be inverted. For this reason in our simulation we used the Matlab function **fminsearch** which is quite robust becasue it uses a simplex search algorithm (Lagarias et al. 1998) and rapidly finds the implied volatility by minimizing the squared difference between the Black-Scholes price (that is found by varying the volatility) and the Heston model call price.

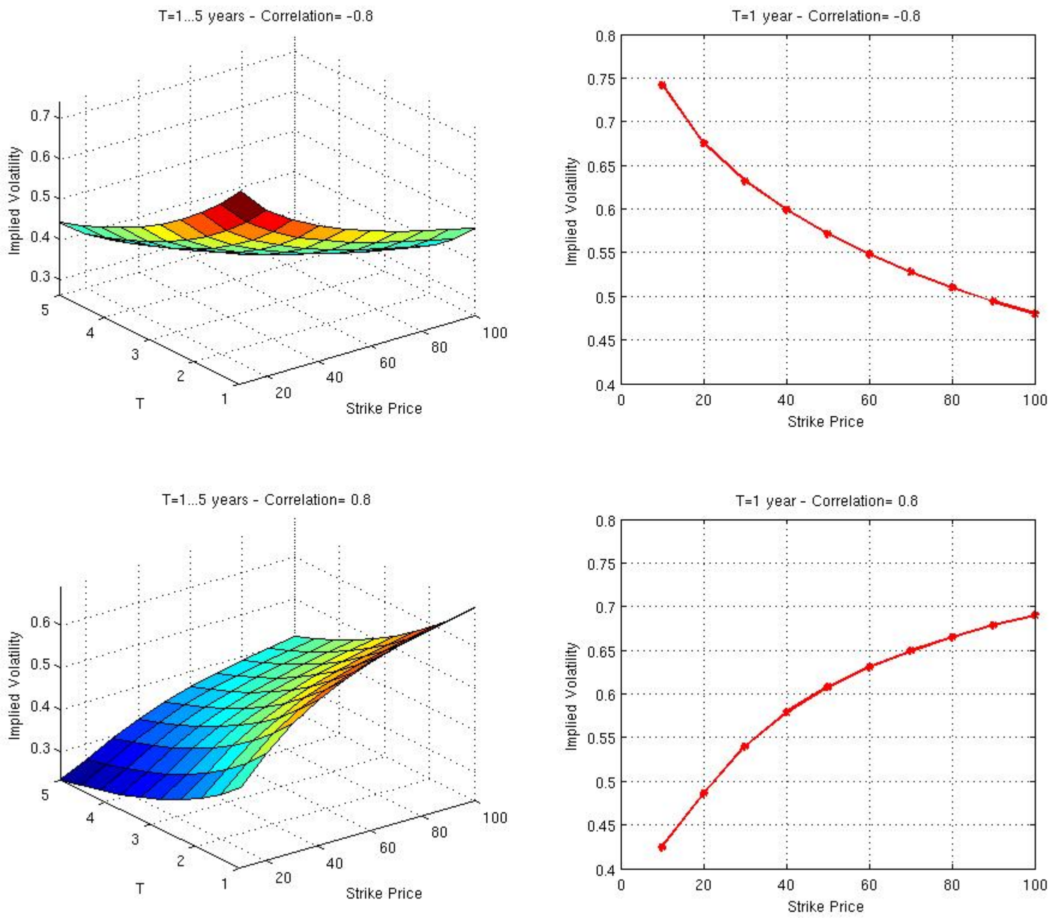


Figure 3.8.: Corresponding volatility implied from Black-Scholes equivalent call price for a fixed strike price, initial asset price and risk-free rate.

For a fixed initial asset price S_0, Figure 3.9 shows that implied volatility is much greater at a low strike price for option deep in the money. A large smirk is observed for a large negative correlation between asset price and volatility. A correlation near zero will flatter the smirk but the implied volatility will still be slightly higher at a low strike price.

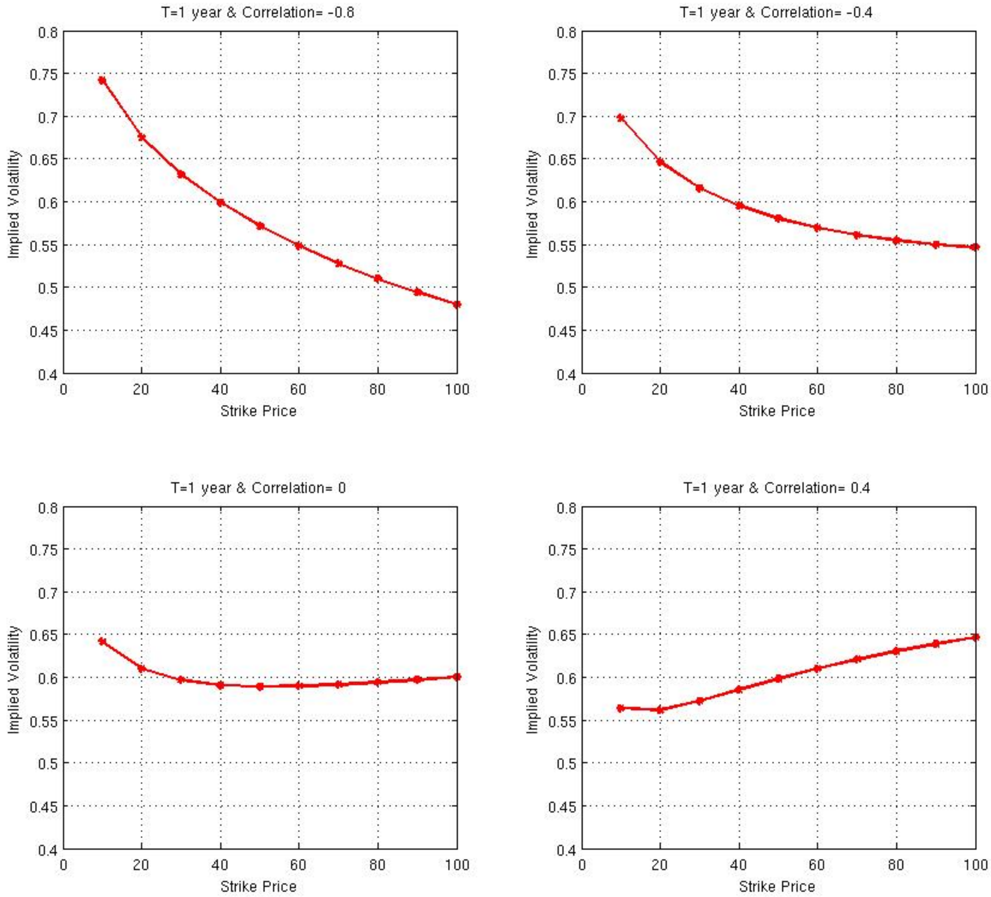


Figure 3.9.: Implied volatility for 4 different correlations $\rho = -0.8, -0.4, 0.0, 0.4$. A correlation near zero will flatter the smirk but the implied volatility will still be slightly higher at a low strike price.

For positive correlation, the implied volatility will be higher for options with a high strike price. Still, stochastic volatility models can only create a steep short-term smile when the volatility of the volatility is large. Another practical feature of the Heston model is the ability for the implied volatility smirk to flatter and decrease for options contracted for a longer expiration time.

Part III.

Numerical Aspects

4. Finite-Difference Methods

4.1. Fundamentals

Finite difference methods (FDM) for option pricing are numerical methods used in mathematical finance for the valuation of options. The finite-difference method gives an approximation of a derivative price evolution on a grid which typically represents time and the underlying asset price but can include and other factors such as rate or volatility. The evolution is dictated by the partial differential equation (PDE) of the asset model and the derivative. The standard method is to start with the known values at expiration and to solve the set of derivative values at the previous time step. In general, finite-difference method is effective for solving derivatives that can be described moving backwards in time such as American options. Additionally, Greeks can be found directly from the gradient in the value of the nodes on the exercising grid. Standard finite difference methods suffer from high computational cost when a derivative depend on several underlying variables and so we have to solve a multidimensional problem.

4.1.1. Difference Approximation

The Taylor series expansion of the first derivative of a function f in the forward and backward direction at a point x gives,

$$\underbrace{f^{'}(x) = \frac{f(x+h)-f(x)}{h} + O(h)}_{forward} \qquad \underbrace{f^{'}(x) = \frac{f(x)-f(x-h)}{h} + O(h)}_{backward}, \quad f \in C^2. \tag{4.1.1}$$

Direct application gives the first derivative of the option price with respect to the stock price about the node (i,j)

$$\underbrace{D^{+}_{\Delta S} f_{i,j} = \frac{f_{i,j+1}-f_{i,j}}{\Delta S}}_{forward} \qquad \underbrace{D^{-}_{\Delta S} f_{i,j} = \frac{f_{i,j}-f_{i,j-1}}{\Delta S}}_{backward} \tag{4.1.2}$$

It is usually best to use an average of the previous two as a symmetric central difference

$$D^0_{\Delta S} f_{i,j} = \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta S} \tag{4.1.3}$$

A similar symmetric difference has the second order derivative of the option price with respect to the stock price about the node (i, j)

$$D^2_{\Delta S} f_{i,j} = \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{(\Delta S)^2} \tag{4.1.4}$$

4.1.2. Finite Difference Grid

The main idea is to examine the evolution of the derivative price across a grid of asset prices and time. Typically, the option expiration is set at time T and the present time is set at t.

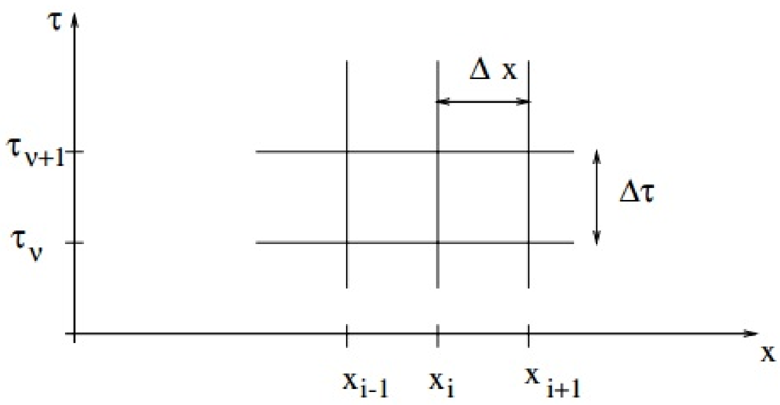


Figure 4.1.: Detail and notations of the grid [Seydel]

The points along time direction are at $N+1$ equally spaced points, with a time step $\Delta\tau = T/N$. Similarly the other axis of the grid consists of $I+1$ equally spaced stock prices, with a stock price step of $\Delta X = X_{max}/I$. This defines a two-dimensional uniform grid as illustrated in Figure 2.

Transforming the (X, τ)-grid to the (S, t)-plane, leads to a nonuniform grid with unequal distances of the grid lines $S = S_i = S_0 e^{X_i}$. The spacing for the log stock price grid

$$X_i = -X_{max}, ..., -2\Delta X, -\Delta X, 0, \Delta X, 2\Delta X, ..., X_{max} \tag{4.1.5}$$

corresponds to a stock price spacing of

$$S_i = S_0 e^{-X_{max}}, ..., S_0 e^{-2\Delta X}, ..., 0, ..., S_0 e^{-2\Delta X}, ..., S_0 e^{X_{max}} \tag{4.1.6}$$

So the value of the derivative $f_{i,j}$ is evaluated as $f_{i,j} = f(\tau = i\Delta\tau, S = j\Delta S)$. The evolution of the backward price moving in time is described by the particular differential equation of the model. The PDE cannot be directly applied to the nodes of the grid. Rather, a Taylor series expansion is used to approximate the derivatives by the values at the current and neighborhoods nodes. This approximation is accomplished by an explicit (forward in time) method, implicit (backward in time) method or a compilation of two the Crank-Nicolson (equally weighted) method.

4.2. European Style Options

4.2.1. Boundary conditions

European call options A first boundary condition for a European call option which is deep in-the-money at S_{max} at any point along the boundary is assumed to pay off $S_T - K$ at expiration time T. The strike price value K is discounted back to time t, and at time t the asset price is $S(t)$ by arbitrage-free pricing. Thus the boundary condition reads

$$c_{i,j=M} = f(t = i\Delta t, S_{max} = M\Delta S) = S_{max}e^{-qt} - e^{-r(T-t)}K. \tag{4.2.1}$$

A second boundary condition is for a deep out-of-the-money European call option at $S = 0$. For any time point along the boundary has no probability to return in-the-money and its value is

$$c_{i,j=0} = f(t = i\Delta t, S_{max} = 0) = 0. \tag{4.2.2}$$

The third boundary condition is when the European call option is valued by definition at time T as

$$c = max(S_T - K, 0) = (S_T - K)^+. \tag{4.2.3}$$

European put options In the same way as previous for the European call options a deep in-the-money European put option at $S = 0$ along the boundary for any time point will be zero for all future time points. At expiration will be worth K. Thus the boundary condition is

$$p_{i,j=0} = f(t = i\Delta t, S = 0) = e^{-r(T-t)}K. \tag{4.2.4}$$

A deep out-of-the-money put at S_{max} for any time point along the boundary which has no probability to to return in-the-money valued by

$$p_{i,j=M} = f(t = i\Delta t, S_{max} = M\Delta S) = 0. \tag{4.2.5}$$

The third boundary condition is when the European put option is valued by definition at time T as

$$p = max(K - S_T, 0) = (K - S_T)^+. \tag{4.2.6}$$

4.2.2. Explicit (Forward in Time) FDM

By applying the forward partial derivative approximations to the Black-Scholes PDE gives

$$\begin{aligned}
&\frac{f_{i+1,j} - f_{i,j}}{\Delta t} + (r-q)j\Delta S \frac{f_{i+1,j+1} - f_{i+1,j-1}}{2\Delta S} + \\
&\frac{1}{2}\sigma^2 (j\Delta S)^2 \frac{f_{i+1,j+1} - 2f_{i+1,j} + f_{i+1,j-1}}{(\Delta S)^2} = r f_{i,j}
\end{aligned}$$

where $S_j = j\Delta S$. Disentangling the current option price and multiplying be a common Δt gives

$$\begin{aligned}
f_{i,j} = f_{i+1,j-1} &\underbrace{\frac{-\frac{1}{2}(r-q)j\Delta t + \frac{1}{2}\sigma^2 j^2 \Delta t}{1 + r\Delta t}}_{a_j} + f_{i+1,j} \underbrace{\frac{1 - \sigma^2 j^2 \Delta t}{1 + r\Delta t}}_{b_j} \\
&+ f_{i+1,j+1} \underbrace{\frac{\frac{1}{2}(r-q)j\Delta t + \frac{1}{2}\sigma^2 j^2 \Delta t}{1 + r\Delta t}}_{c_j}
\end{aligned}$$

or with in a short-hand notation

$$f_{i,j} = a_j f_{i+1,j-1} + b_j f_{i+1,j} + c_j f_{i+1,j+1} \tag{4.2.7}$$

4.2.3. Implicit (Backward in Time) FDM

By applying the forward partial derivative approximations to the Black-Scholes PDE gives

$$\begin{aligned}
&\frac{f_{i+1,j} - f_{i,j}}{\Delta t} + (r-q)j\Delta S \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta S} \\
&+ \frac{1}{2}\sigma^2 (j\Delta S)^2 \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{(\Delta S)^2} = r f_{i,j}
\end{aligned}$$

Rearranging gives

$$\underbrace{\left(\frac{1}{2}(r-q)j\Delta t - \frac{1}{2}\sigma^2 j^2 \Delta t\right)}_{a_j} f_{i,j-1} + \underbrace{(1 + r\Delta t + \sigma^2 j^2 \Delta t)}_{b_j} f_{i,j} + \underbrace{\left(-\frac{1}{2}(r-q)j\Delta t - \frac{1}{2}\sigma^2 j^2 \Delta t\right)}_{c_j} f_{i,j+1} = f_{i+1,j}$$

This can be expressed in matrix form as

$$\underbrace{\begin{bmatrix} b_0 & c_0 & 0 & \cdots & 0 \\ a_1 & b_1 & c_1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & a_{M-1} & b_{M-1} & c_{M-1} \\ 0 & 0 & 0 & a_M & b_M \end{bmatrix}}_{M} \begin{bmatrix} f_{i,0} \\ f_{i,1} \\ \vdots \\ f_{i,M-1} \\ f_{i,M} \end{bmatrix} = \begin{bmatrix} f_{i+1,0} \\ f_{i+1,1} \\ \vdots \\ f_{i+1,M-1} \\ f_{i+1,M} \end{bmatrix} \tag{4.2.8}$$

The implicit method relaxes the Courant [CFL] stability limit, and thus the number of required time steps is usually less. The value of the options at the previous time step is repeatedly solved until the present time is reached. For a model without stochastic volatility, the matrix M does not vary in time and can be calculated just once.

4.2.4. The Crank-Nicolson Method

The Crank-Nicolson method attempts to improve the stability and convergence averaging explicit and implicit techniques. The CN method, or trapezoidal method, relies in symmetric weighting at the half time step and given by

$$f_{i+\frac{1}{2},j} = \frac{1}{2}(f_{i,j} + f_{i+1,j}) \tag{4.2.9}$$

Applying to the Black-Scholes finite differential equation (3.2.27) we take

$$\frac{f_{i+1,j} - f_{i,j}}{\Delta t} + (r-q)j\Delta S \frac{f_{i+\frac{1}{2},j+1} - f_{i+\frac{1}{2},j-1}}{2\Delta S} + \frac{1}{2}\sigma^2 (j\Delta S)^2 \frac{f_{i+\frac{1}{2},j+1} - 2f_{i+\frac{1}{2},j} + f_{i+\frac{1}{2},j-1}}{(\Delta S)^2} = r f_{i+\frac{1}{2},j}$$

Separating the explicit and implicit terms gives

$$\frac{f_{i+1,j}-f_{i,j}}{\Delta t}+(r-q)j\Delta S\frac{\frac{1}{2}(f_{i,j+1}+f_{i+1,j+1})-\frac{1}{2}(f_{i,j-1}+f_{i+1,j-1})}{2\Delta S}$$
$$+\frac{1}{2}\sigma^2(j\Delta S)j^2\frac{\frac{1}{2}(f_{i,j+1}+f_{i+1,j+1})-(f_{i,j}+f_{i+1,j})+\frac{1}{2}(f_{i,j-1}+f_{i+1,j-1})}{(\Delta S)^2}$$
$$=r\frac{1}{2}(f_{i,j}+f_{i+1,j})$$

Canceling and rearranging gives

$$\frac{f_{i+1,j}-f_{i,j}}{\Delta t}+\frac{1}{4}(r-q)j(f_{i,j+1}-f_{i+1,j-1})$$
$$+\frac{1}{4}(r-q)j(f_{i+1,j+1}-f_{i+1,j-1})$$
$$+\frac{1}{2}j^2\sigma^2(\frac{1}{2}f_{i,j+1}-f_{i,j}+\frac{1}{2}f_{i,j-1})$$
$$+\frac{1}{2}j^2\sigma^2(\frac{1}{2}f_{i+1,j+1}-f_{i+1,j}+\frac{1}{2}f_{i+1,j-1})$$
$$=(r\frac{1}{2}f_{i,j}+r\frac{1}{2}f_{i+1,j})$$

Arranging into the format of Brandimarte [Bra2006], we get

$$\frac{1}{4}\Delta t\{-j^2\sigma^2+(r-q)j\}f_{i,j-1}+\left\{1+\Delta t\frac{1}{2}(r+j^2\sigma^2)\right\}f_{i,j}$$
$$-\frac{1}{4}\Delta t\left\{j^2\sigma^2+(r-q)j\right\}f_{i,j+1}$$
$$=\frac{1}{4}\Delta t\left\{j^2\sigma^2-(r-q)j\right\}f_{i+1,j-1}+\left\{1-\Delta t\frac{1}{2}(r+j^2\sigma^2)\right\}f_{i+1,j}$$
$$+\frac{1}{4}\Delta t\left\{j^2\sigma^2+(r-q)j\right\}f_{i+1,j+1}$$

We can rewrite the previous formula in a sorter form as

$$-a_jf_{i,j-1}+(1-\beta_j)f_{i,j}-\gamma_jf_{i,j+1}=a_jf_{i+1,j+1}+(1+\beta_j)f_{i+1,j}+\gamma_jf_{i+1,j+1} \quad (4.2.10)$$

where the coefficient parameters are

$$
\begin{aligned}
a_j &= \frac{1}{4}\Delta t\{-j^2\sigma^2 + (r-q)j\} \\
\beta_j &= -\Delta t\frac{1}{2}(r + j^2\sigma^2) \\
\gamma_j &= \frac{1}{4}\Delta t\left\{j^2\sigma^2 + (r-q)j\right\}
\end{aligned}
$$

In matrix notation the previous formula gives $M_1 f_i = M_2 f_{i+1}$

$$
\overbrace{\begin{bmatrix}
1-\beta_1 & -\gamma_1 & 0 & \cdots & 0 \\
-a_2 & 1-\beta_2 & -\gamma_2 & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & a_{M-2} & 1+\beta_{M-2} & -\gamma_{M-2} \\
0 & 0 & 0 & -a_{M-1} & 1-\beta_{M-1}
\end{bmatrix}}^{M1}
\begin{bmatrix} f_{i,1} \\ f_{i,2} \\ \vdots \\ f_{i,M-2} \\ f_{i,M-1} \end{bmatrix}
$$

$$
= \underbrace{\begin{bmatrix}
1+\beta_1 & \gamma_1 & 0 & \cdots & 0 \\
a_2 & 1+\beta_2 & \gamma_2 & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & a_{M-2} & 1+\beta_{M-2} & \gamma_{M-2} \\
0 & 0 & 0 & a_{M-1} & 1+\beta_{M-1}
\end{bmatrix}}_{M2}
\begin{bmatrix} f_{i+1,1} \\ f_{i+1,2} \\ \vdots \\ f_{i+1,M-2} \\ f_{i+1,M-1} \end{bmatrix}
$$

In quasi-matrix format, the boundary values are added to the interior matrix terms by

$$
\begin{aligned}
M_1 f_i = M_2 f_{i+1} - \begin{bmatrix} -a_1 f_{i,0} \\ 0 \\ \vdots \\ 0 \\ -\gamma_{M-1} f_{i,M} \end{bmatrix} + \begin{bmatrix} a_1 f_{i+1,0} \\ 0 \\ \vdots \\ 0 \\ \gamma_{M-1} f_{i+1,M} \end{bmatrix} \\
\underbrace{= M_2 f_{i+1} + \begin{bmatrix} a_1(f_{i+1,0} + f_{i,0}) \\ 0 \\ \vdots \\ 0 \\ \gamma_{M-1}(f_{i+1,M} + f_{i,M}) \end{bmatrix}}_{r_{i+1}}
\end{aligned}
$$

The simplest technique to find the f_i values is matrix left division, (in MATLAB notation $f_i = M_1 \backslash r_{i+1}$) repeated at each time step i. More elegant is a precalculation of the LU decomposition and then a two-step matrix left division at each time step ((in MATLAB notation $f_i = U \backslash (L \backslash r_{i+1})$). These approaches are valuable to give a European option value but cannot be applied to calculate American style options. The main issue is that an American option requires the option value to be compared to its intrinsic value via the maximum function at each step and node. To tackle this problem many different numerical techniques have been developed.

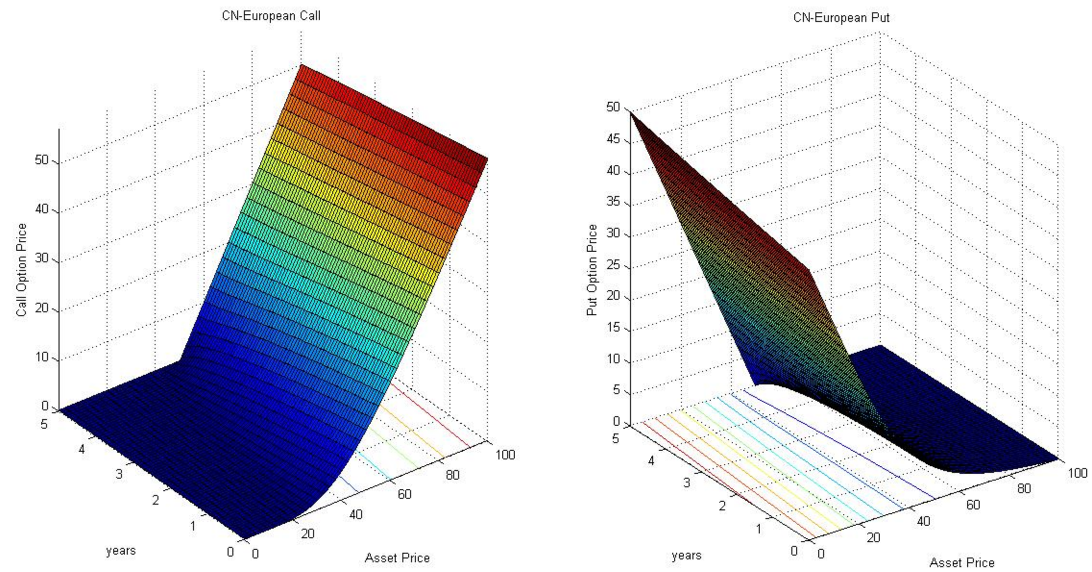


Figure 4.2.: European call and put option price via Crank-Nicolson on a PDE grid for a non-dividend-paying commodity

4.3. American Style Options

4.3.1. Successive Over relation Technique

The implicit or Crank-Nicolson technique concurrently calculates all the nodes at the current time layer. Thus, if one option node value converts to its intrinsic value, then there is no mechanism to spread this new information to the other nodes.

A preferred technique approaches the correct set of values within a small tolerance via a successive over relation (SOR) [Cryer]. In SOR, the old set of values are updated by the difference with the new set of values multiplied by a dumping function ω. A generic matrix can be divided into the strictly lower triangular matrix L, strictly upper triangular matrix U, and the diagonal D. Thus, matrix equation

$Mx = r$ can be written as

$$\begin{aligned} Dx + Lx + Ux &= r \\ \omega Dx + \omega Lx + \omega Ux &= \omega r \\ Dx - Dx + \omega Dx + \omega Lx + \omega Ux &= \omega r \\ [D + \omega L]x + [\omega U + (\omega - 1)D]x &= \omega r \\ x &= [D + \omega L]^{-1} \{\omega r - [\omega U + (\omega - 1)D]x\}. \end{aligned}$$

The goal is to iteratively update the value $x^{(k+1)}$ from the previous iteration $x^{(k)}$

$$x^{(k+1)} = [D + \omega L]^{-1} \left\{\omega r - [\omega U + (\omega - 1)D]x^{(k)}\right\}. \quad (4.3.1)$$

As $D+\omega L$ is triangular in form, the update simplifies to this node-by-node sequential update as

$$x^{(k+1)} = (1 - \omega)x_i^{(k)} + \frac{\omega}{a_{ii}} \left(r_i - \sum_{j>i} a_{ij}x_j^{(k)} - \sum_{j<i} a_{ij}x_j^{(k+1)} \right). \quad (4.3.2)$$

The damping function must be in the range $0 < \omega < 2$. It is not possible to predict the optimal damping parameter (except the case of special matrices), but the range $\omega = 1.2 - 1.5$ is usually satisfactory.

4.3.2. Crank-Nicolson Scheme For American Options

The Crank-Nicolson scheme for a Black-Scholes option PDE, with the known option values contained in r_{i+1}, is

$$M_1 f_i = r_{i+1} \quad (4.3.3)$$

with the triadiagonal matrix M_1

$$M1 = \begin{bmatrix} 1-\beta_1 & -\gamma_1 & 0 & \cdots & 0 \\ -a_2 & 1-\beta_2 & -\gamma_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & a_{M-2} & 1+\beta_{M-2} & -\gamma_{M-2} \\ 0 & 0 & 0 & -a_{M-1} & 1-\beta_{M-1} \end{bmatrix} \quad (4.3.4)$$

which allows a simplification via the SOR scheme to

$$f_j^{(k+1)} = f_j^{(k)} + \frac{\omega}{(1-\beta_1)}(r_{i+1} + \gamma_j f_j^{(k)} - (1-\beta_j)f_j^{(k)} + a_j f_{j-1}^{(k+1)}). \quad (4.3.5)$$

4.3.3. American Options as Free Boundary Problems

The value of an American option can never be smaller than the value of a European option

$$V^{Am} \geq V^{Eur}. \tag{4.3.6}$$

In addition, an American option has at least the value of the payoff. So we have elementary lower bounds for the value of American options.

4.3.3.1. Early-Exercise Curve

A European option can have a value that is smaller than the payoff. This can not happen with American options. If for instance an American put would have a value $V_P^{Am} < (K - S)^+$, one would simultaneously purchase the asset and the put, and exercise immediately. An analogous arbitrage argument implies that for an American call the situation $V_C^{Am} < (S - K)^+$ can not prevail. Therefore the following inequalities

$$\begin{aligned} V_P^{Am} &\geq (K - S)^+, \\ V_C^{Am} &\geq (S - K)^+, \end{aligned} \tag{4.3.7}$$

hold for all (S, t). This result agrees also with our simulation.

4.4. Option Greeks

The computation of the Greeks plays an important role in trading strategies and the implied risk. The valuation of certain option Greeks is intrinsically calculated by the PDE grid. The grid based techniques used above calculated the value of the option price for a several underlying asset prices over a range of time. Delta is the rate of change of the option price with respect to the underlying asset price. A central difference scheme extracts Delta from the PDE grid as

$$\Delta_c = D_{\Delta S}^0 c_{i,j} = \frac{c_{i,j+1} - c_{i,j-1}}{2\Delta S} \tag{4.4.1}$$

$$\Delta_p = D_{\Delta S}^0 p_{i,j} = \frac{p_{i,j+1} - p_{i,j-1}}{2\Delta S} \tag{4.4.2}$$

Similarly, the Greek Gamma is found by

$$D_{\Delta S}^2 c_{i,j} = \frac{c_{i,j+1} + 2c_{i,j} - c_{i,j-1}}{\Delta S^2} \tag{4.4.3}$$

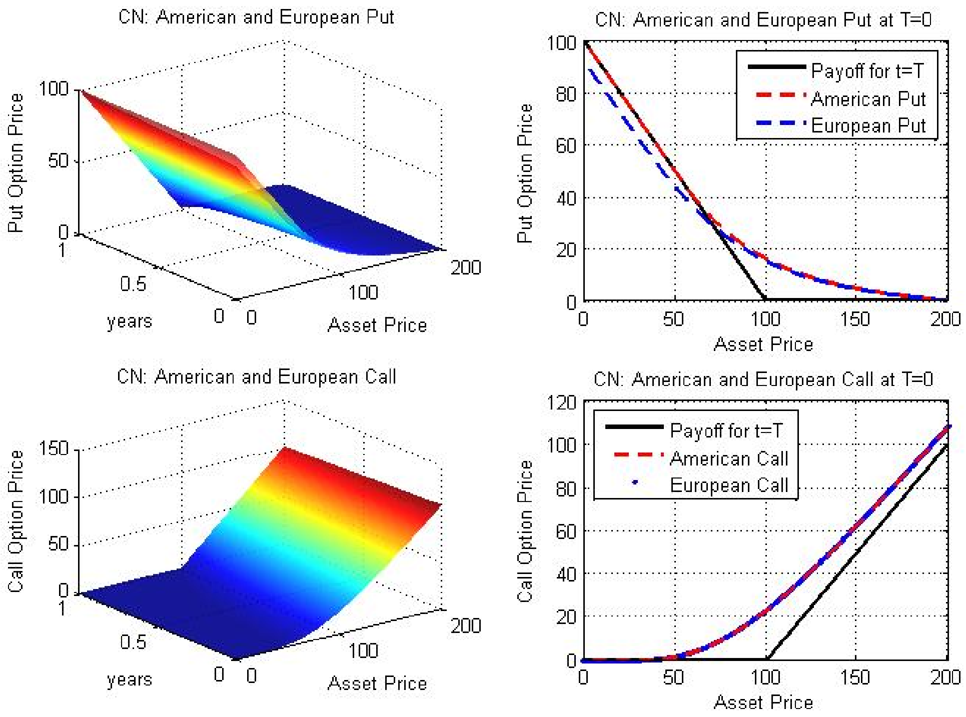


Figure 4.3.: European and American call option for $t < T$. $K = 100$, $r = 0.03$, $\sigma = 0.3$ and $T = 1$

$$D^2_{\Delta S} p_{i,j} = \frac{p_{i,j+1} + 2p_{i,j} - p_{i,j-1}}{\Delta S} \tag{4.4.4}$$

Adding a stochastic volatility process to augment the asset price model is known to replicate many of the features observed in market data. Equity options display implied volatility that indicate a negative correlation between volatility and asset price. The next section introduces an additional dimension on the PDE grid to describe the stochastic volatility.

4.5. Heston - Multidimensional PDE

Several models have been proposed that introduce a second stochastic factor to the particular model. To introduce a second factor requires adding an additional dimension to the finite difference approach. The Heston model is particularly popular among the stochastic volatility models. The Heston PDE as we show is given by

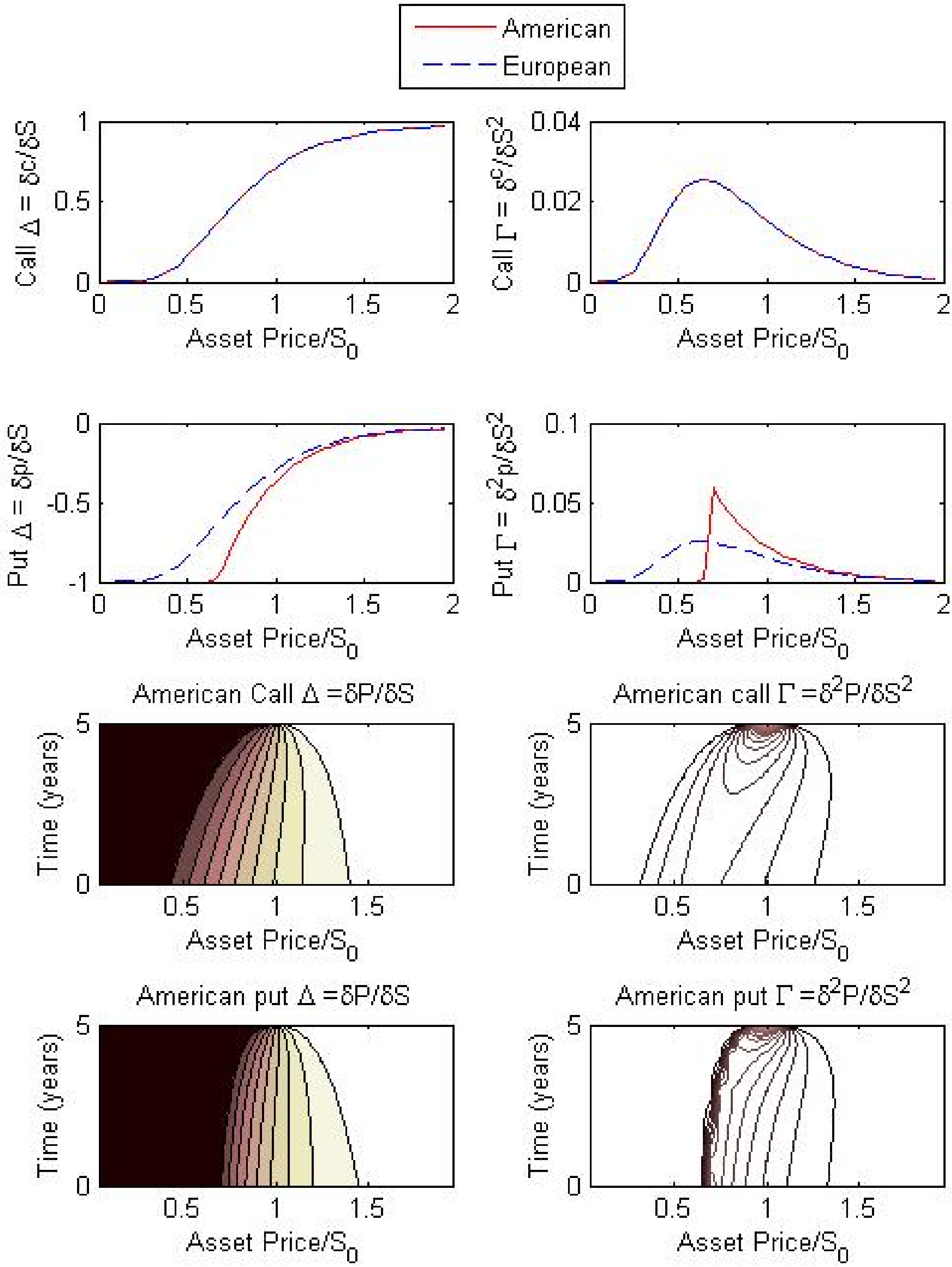


Figure 4.4.: A comparison of Delta and Gamma for a European (via matrix division) and American (via SOR) for option values on a PDE grid

$$\left\{\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}uS^2 + \frac{1}{2}\frac{\partial^2 V}{\partial u^2}\sigma^2 u + \sigma u \rho S \frac{\partial^2 V}{\partial u \partial S}\right\}$$
$$- rV + rS\frac{\partial V}{\partial S} + \{k(\theta - u_t) - \lambda(S,u,t)\}\frac{\partial V}{\partial u} = 0$$

with $t \in [t_0, T]$, $S \in [S_0, S_{max}]$, $u \in [0, u_{max}]$. The Heston model can be rewritten as [Galiotos08]

$$\frac{\partial V}{\partial t} + AV - rV = 0 \tag{4.5.1}$$

where A is a generator of the Heston model

$$A = \underbrace{rS\frac{\partial}{\partial S} + \frac{1}{2}S^2 u \frac{\partial^2}{\partial u^2}}_{Black-Scholes} +$$
$$\underbrace{\{k(\theta - u_t) - \lambda(S,u,t)\}\frac{\partial}{\partial u} + \frac{1}{2}\frac{\partial^2}{\partial u^2}\sigma^2 u + \sigma u \rho S \frac{\partial^2}{\partial u \partial S}}_{Stochastic\,volatility}$$

4.5.1. Explicit Heston Finite Difference Approach

The finite difference scheme takes the known intrinsic values of a call or put at contract expiration and repeatedly marches back in time, calculating the option value at every node at each time step. The explicit approach calculates the option values at the current time step $t_n = ndt$ only from known option values at the previously calculated (forward in time, $t_{n+1} = (n+1)dt$) set of nodes. The Heston model is stochastic in variance and asset price, which necessitates a two-dimensional finite grid. Substituting the appropriate partial differentiations into (4.5.1) gives the explicit finite difference scheme as

$$\frac{V_{i,j}^{n-1} - V_{i,j}^{n}}{\Delta t} = \begin{bmatrix} (S_i)^2 V_j^n \frac{V_{i+1,j}^n - 2V_{i,j}^n + V_{i-1,j}^n}{2(\Delta S)^2} \\ +\rho\sigma S_i V_j^n \frac{V_{i+1,j+1}^n + V_{i-1,j-1}^n - V_{i-1,j+1}^n - V_{i+1,j-1}^n}{4\Delta S \Delta u} \\ +\sigma^2 V_j^n \frac{V_{i,j+1}^n - 2V_{i,j}^n + V_{i,j-1}^n}{2(\Delta u)^2} + rS_i \frac{V_{i+1,j}^n - V_{i-1,j}^n}{2\Delta S} \\ +\{k(\theta - u_t) - \lambda\}\frac{V_{i,j+1}^n - V_{i,j-1}^n}{2\Delta u} - rV_{i,j}^n \end{bmatrix} \tag{4.5.2}$$

Rearranging gives the dynamic equation for the explicit finite difference scheme as

$$\begin{aligned} V_{i,j}^{n-1} = {} & A_{i,j}^n V_{i,j}^n + B_{i,j}^n\{V_{i+1,j-1} + V_{i-1,j+1} - V_{i-1,j+1} - V_{i+1,j-1}\} \\ & + C_{i,j}^n V_{i-1,j}^n + D_{i,j}^n V_{i+1,j}^n + E_{i,j}^n V_{i,j-1}^n + F_{i,j}^n V_{i,j+1}^n \end{aligned} \tag{4.5.3}$$

where the coefficients are given as [Lin2008]

$$\begin{aligned}
A_{i,j}^n &= 1 - i^2 u_j \Delta t - \tfrac{\sigma^2 j \Delta t}{\Delta u} - r\Delta t \\
B_{i,j}^n &= \tfrac{\rho \sigma i j}{4} \Delta t \\
C_{i,j}^n &= \left(\tfrac{i^2 u_j}{2} - \tfrac{ri}{2}\right) \Delta t \\
D_{i,j}^n &= \left(\tfrac{i^2 u_j}{2} + \tfrac{ri}{2}\right) \Delta t \\
E_{i,j}^n &= \left(\tfrac{\sigma^2 j}{2\Delta u} - \tfrac{k(\theta - u_t) - \lambda}{2\Delta u}\right) \Delta t \\
F_{i,j}^n &= \left(\tfrac{\sigma^2 j}{2\Delta u} + \tfrac{k(\theta - u_t) - \lambda}{2\Delta u}\right) \Delta t
\end{aligned} \tag{4.5.4}$$

with $S_i = i\Delta S$ and $u_j = j\Delta u$.

4.5.1.1. Explicit Stability limit

The Δt time step must be sufficiently small relative to the other parameters to maintain stability. Specifically, the parameters A, C, D, E and F must be positive.

The parameter A requires

$$\begin{aligned}
A_{i,j}^n &= 1 - i^2 u_j \Delta t - \frac{\sigma^2 j \Delta t}{\Delta u} - r\Delta t \geq 0 \rightarrow \\
\Delta t &\leq \frac{1}{i^2 u_j \Delta t - \frac{\sigma^2 j}{\Delta u} + r} = \frac{1}{i^2 u_j - \frac{J\sigma^2 j}{u_{max}} + r}
\end{aligned}$$

as $\Delta u = \frac{u_{max}}{J}$. Considering only the largest variance value u_{max} and largest i-value (of I) gives the stability limit as

$$\Delta t \leq \frac{1}{I^2 u_{max} + J\sigma^2 + r} \tag{4.5.5}$$

The parameter C requires

$$C_{i,j}^n = \left(\frac{i^2 u_j}{2} - \frac{ri}{2}\right) \Delta t \geq 0 \rightarrow i \geq \frac{r}{u_j} \tag{4.5.6}$$

The parameter E requires

$$E_{i,j}^n = \left(\frac{\sigma^2 j}{2\Delta u} - \frac{k(\theta - u_t) - \lambda}{2\Delta u}\right) \Delta t \geq 0 \rightarrow i \geq \frac{k(\theta - u_t) - \lambda}{\sigma^2} \tag{4.5.7}$$

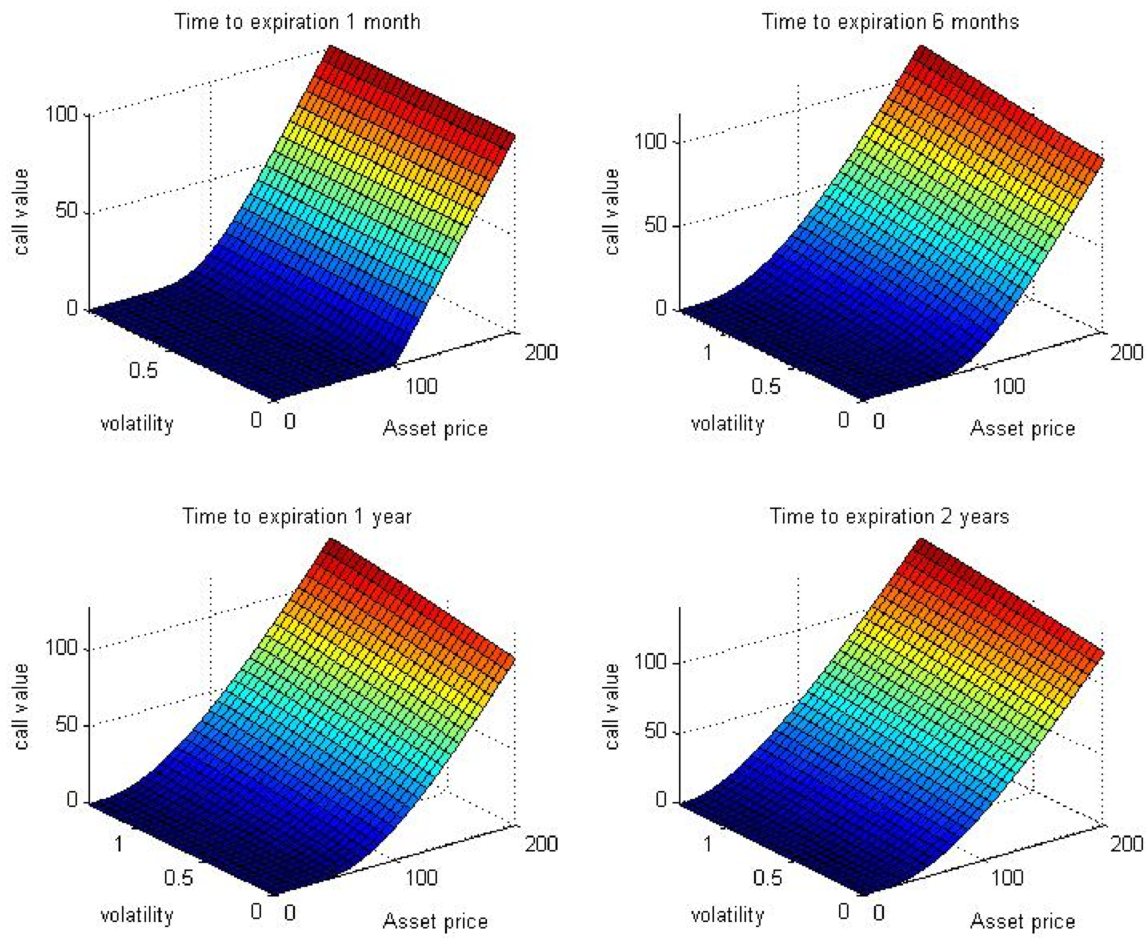


Figure 4.5.: Explicit finite difference calculation for the Heston model on a 2-dimensional grid of volatility and asset price for a European call option contracted for different expirations.

4.5.1.2. Heston PDE Boundary condition

The boundary conditions in the Heston model at expiration, or at the minimum or maximum asset price, is similar to the boundary conditions discussed earlier on this chapter for the Black-Scholes model (model without stochastic volatility). For review, the terminal condition at expiration for a call option is $c(S, u, t = T) = max(S_T - K, 0)$ or for a put option is $p(S, u, t = T) = max(K - S_T, 0)$. The plane of nodes that correspond to an asset price of zero is assumed to have a call option value of zero, $c(S = 0, u, t) = 0$. This type of constant boundary condition is also known as a Dirichlet boundary condition.

The plain of nodes that correspond to the maximum asset price can use either a constant-value Dirichlet or constant-first-derivative Neumann boundary condition. The Dirichlet boundary condition assumes that the call option value is equal to its intrinsic value, $c(S = S_{max}, u, t) = S_{max} - K$. The Neumann boundary condition assumes the option value near the boundary grows linearly in S,

$$\frac{\partial c(S = S_{max}, u, t)}{\partial S} = 1, \tag{4.5.8}$$

which is expressed in discrete form as

$$c(S = I\Delta S, u, t) = c(S = (I-1)\Delta S, u, t) + \Delta S. \tag{4.5.9}$$

The new issue is using a discrete version of the stochastic volatility Heston model is how to deal with the boundary at the maximum and minimum volatility. Logically, the option value increases with increasing volatility. The increase tends to level out at very high variance. Therefore, at the plane of the maximum variance, the rate of change in option price with respect to the volatility is set to zero

$$\frac{\partial c(S, u = u_{max}, t)}{\partial u} = 0, \tag{4.5.10}$$

which is expressed in discrete form as

$$c(S, u = J\Delta u, t) = c(S, u = (J-1)\Delta u, t). \tag{4.5.11}$$

The choice at the minimum volatility requires examining the Heston PDE with $u = 0$, that is

$$\frac{\partial c}{\partial t} = -rc + rS\frac{\partial c}{\partial S} + \{k\theta - \lambda\}\frac{\partial c}{\partial u}. \tag{4.5.12}$$

The corresponding explicit finite difference equation at $j = 0$ is

$$\frac{c_{i,0}^{n-1} - c_{i,0}^{n}}{\Delta t} = rS_i\frac{c_{i+1,0}^{n} - c_{i-1,0}^{n}}{2\Delta S} + (k\theta - \lambda)\frac{c_{i,1}^{n} - c_{i,0}^{n}}{\Delta u} - rc_{i,0}^{n}, \tag{4.5.13}$$

which yields

$$\begin{aligned} c_{i,0} = rS_i \tfrac{c_{i+1,0}^{n} - c_{i-1,0}^{n}}{2\Delta S}\Delta t + \{k\theta - \lambda\}\tfrac{\Delta t}{\Delta u}c_{i,1}^{n} \\ +(1 - r\Delta t - \{k\theta - \lambda\}\tfrac{\Delta t}{\Delta u})c_{i,0}^{n}. \end{aligned} \tag{4.5.14}$$

Evaluating an American style call or put option by the explicit technique is fairly straightforward, as it only requires comparison of all the nodes to their intrinsic value at each time step.

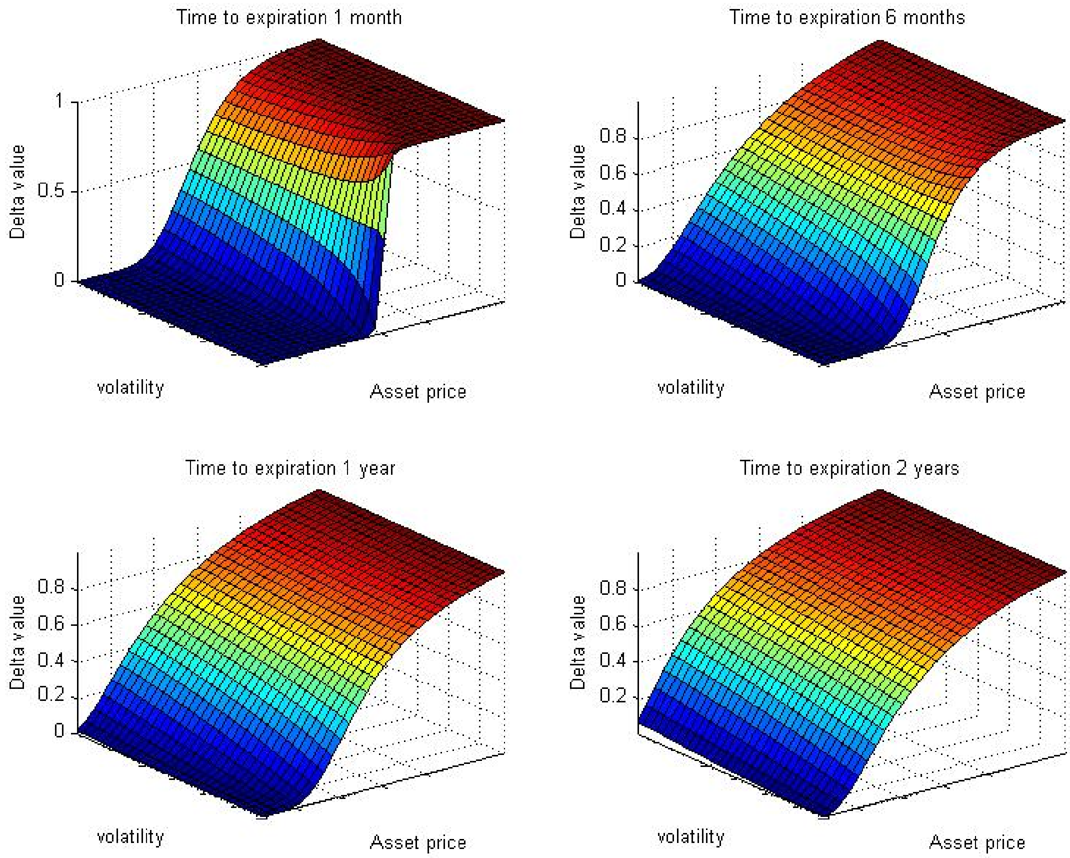


Figure 4.6.: Explicit finite difference calculation for the Heston model on a 2-dimensional grid of volatility and asset price for the delta of a European call option contracted for different expirations.

4.5.1.3. Numerical Simulation

We perform numerical simulations with a small time step $\Delta t = T/1000$ to obtain a sufficiently accurate approximation of the option value c in the four cases of parameter sets given by Table 4.1. Observe that in three of the four cases there is a substancial correlation factor ρ. Only in Case 4 the correlation factor is relatively small.

Case 1 has been taken from Albrecher et. al. [Alb07], where we have chosen $T = 1$. Case 2 comes from Bloomberg [Blo05]. A special feature of this parameter set is that σ is close to zero, which implies that the Heston PDE is convection-dominated in the value-direction. Values of r and T were not specified in Bloomberg and have been chosen separately. Case 3 has been taken from Schoutens et. al. [Sch04]. Here the Feller condition is satisfied in the limit case. Finally, Case 4 stems from Winkler et. al. [Winkler02].

	Case 1	**Case 2**	**Case 3**	**Case 4**
κ	1.5	3	0.6067	2.5
θ	0.04	0.12	0.0707	0.06
σ	0.3	0.04	0.2928	0.5
ρ	-0.9	0.6	-0.7571	-0.1
r	0.025	0.01	0.03	0.0507
T	1	1	3	0.25
K	100	100	100	100

Table 4.1.: Parameters for the Heston model and European call options

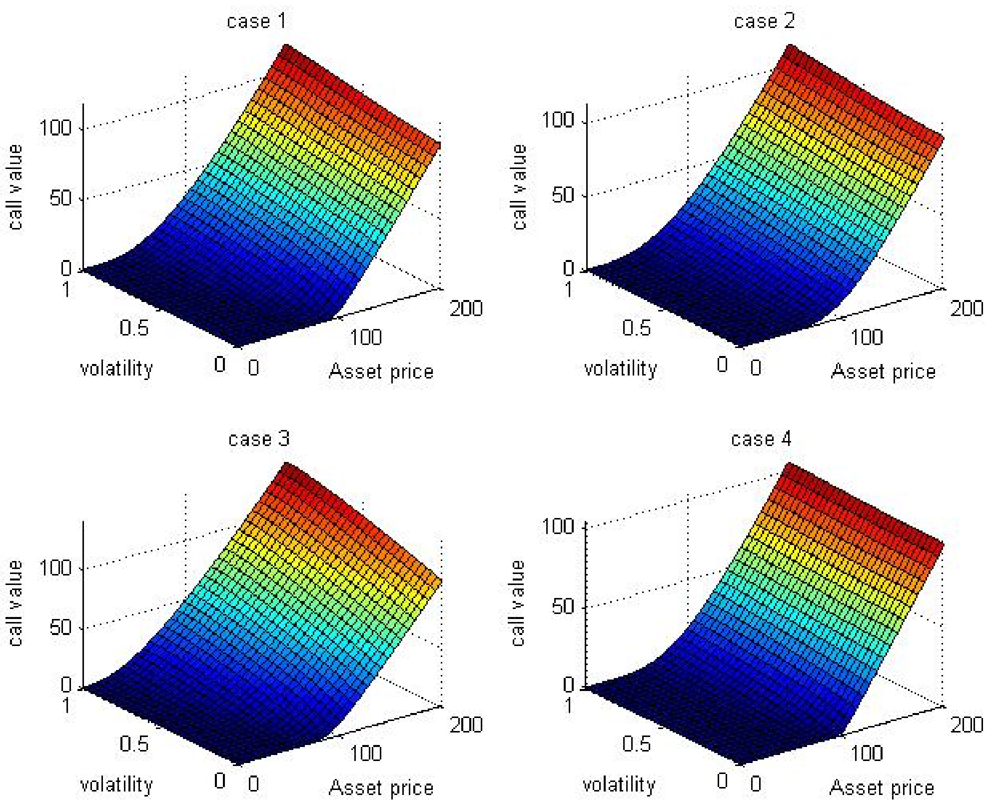


Figure 4.7.: Heston model European call option value in the four cases given by Table 4.1.

We employ an implementation of Heston's semi-analytical formula with the parameter set of Case1.

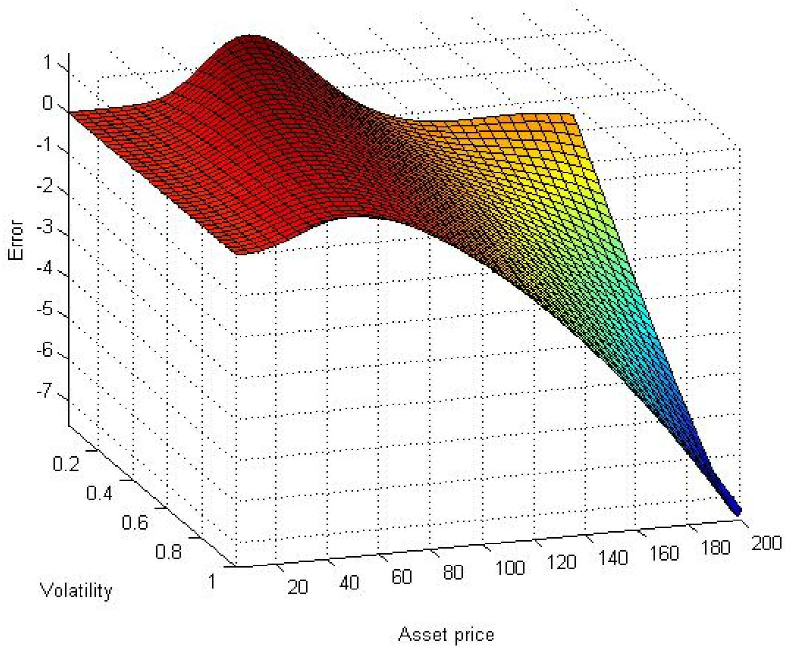


Figure 4.8.: Surface of the value error between Heston explicit model and semi-analytical formula

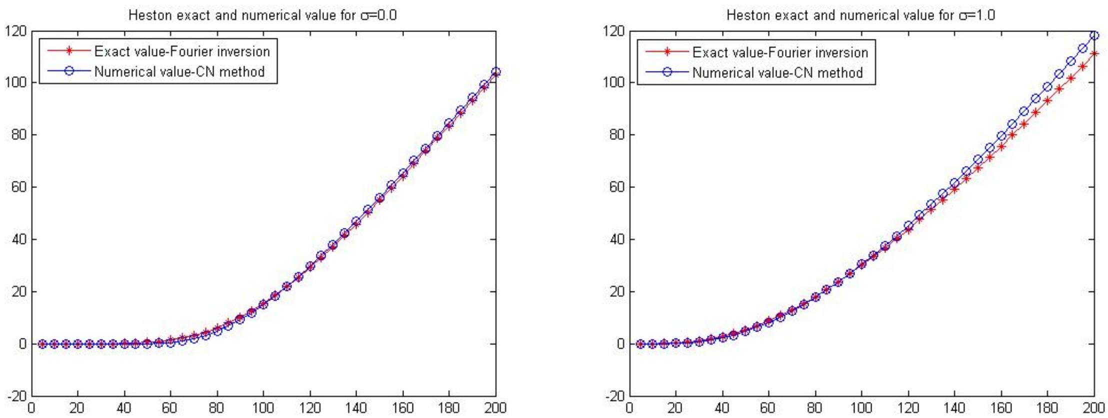


Figure 4.9.: (left) Option value for $\sigma = 0$ (right) option value for $\sigma = 1$.

From the Figure 4.8 and Figure 4.9, we can observe that the error is increased when the volatility increases and we are deeper in-the-money (ITM).

4.5.2. Implicit Heston Finite Difference Approach

The implicit approach calculates the option values at the current time step $t_n = ndt$ in a coupled approach. Only known option value at the previously calculated (forward in time, $t_{n+1} = (n+1)dt$) set of nodes is used in the calculation. The implicit finite difference scheme is

$$\frac{V_{i,j}^n - V_{i,j}^{n+1}}{\Delta t} = \begin{bmatrix} (S_i)^2 V_j^n \frac{V_{i+1,j}^n - 2V_{i,j}^n + V_{i-1,j}^n}{2(\Delta S)^2} \\ +\rho\sigma S_i V_j^n \frac{V_{i+1,j+1}^n + V_{i-1,j-1}^n - V_{i-1,j+1}^n - V_{i+1,j-1}^n}{4\Delta S \Delta u} \\ +\sigma^2 V_j^n \frac{V_{i,j+1}^n - 2V_{i,j}^n + V_{i,j-1}^n}{2(\Delta u)^2} + rS_i \frac{V_{i+1,j}^n - V_{i-1,j}^n}{2\Delta S} \\ +(k(\theta - u_t) - \lambda)\frac{V_{i,j+1}^n - V_{i,j-1}^n}{2\Delta u} - rV_{i,j}^n \end{bmatrix} \tag{4.5.15}$$

Rearranging gives the dynamic equation for the implicit finite difference scheme as

$$\begin{aligned} V_{i,j}^{n+1} = a_{i,j}^n V_{i,j}^n + b_{i,j}^n \{V_{i+1,j-1} + V_{i-1,j+1} - V_{i-1,j+1} - V_{i+1,j-1}\} \\ +c_{i,j}^n V_{i-1,j}^n + d_{i,j}^n V_{i+1,j}^n + e_{i,j}^n V_{i,j-1}^n + f_{i,j}^n V_{i,j+1}^n \end{aligned} \tag{4.5.16}$$

where [Lin2008]

$$\begin{aligned} a_{i,j}^n &= 1 + i^2 u_j \Delta t + \tfrac{\sigma^2 j \Delta t}{\Delta u} + r\Delta t \\ b_{i,j}^n &= -\tfrac{\rho\sigma ij}{4}\Delta t \\ c_{i,j}^n &= -\left(\tfrac{i^2 u_j}{2} - \tfrac{ri}{2}\right)\Delta t \\ d_{i,j}^n &= -\left(\tfrac{i^2 u_j}{2} + \tfrac{ri}{2}\right)\Delta t \\ e_{i,j}^n &= -\left(\tfrac{\sigma^2 j}{2\Delta u} - \tfrac{k(\theta - u_t) - \lambda}{2\Delta u}\right)\Delta t \\ f_{i,j}^n &= -\left(\tfrac{\sigma^2 j}{2\Delta u} + \tfrac{k(\theta - u_t) - \lambda}{2\Delta u}\right)\Delta t \end{aligned} \tag{4.5.17}$$

If we compare the implicit with the explicit coefficients we will see that are equal in magnitude but opposite in sign, except the coefficient $a_{i,j}^n = -A_{i,j}^n + 2$.

When we studied the impicit technique in the one-dimensional model (Black-Scholes), it was discussed that the implicit method relaxes the Courant–Friedrichs–Lewy condition [CFL] that constrains the explcit technique. As a consequence, the time step must be less than a certain time in many explicit time-marching computer simulations, otherwise the simulation will produce incorrect results. In other words, the implicit technique allows a larger time interval and fewer total time steps. Nevertheless, the computational burden can be increased, as some type of inversion is necessary. In the one-dimensional, the option value caculated in a stepwise manner at successive time steps. The same technique can be used and here. This technique can accelerate convergence to the American option value within some small error tolerance.

We run simulation for explicit scheme with $T = 1$ and $dt = 1/1000$. This time step is needed in our simulation case to be stable. The computational time is a

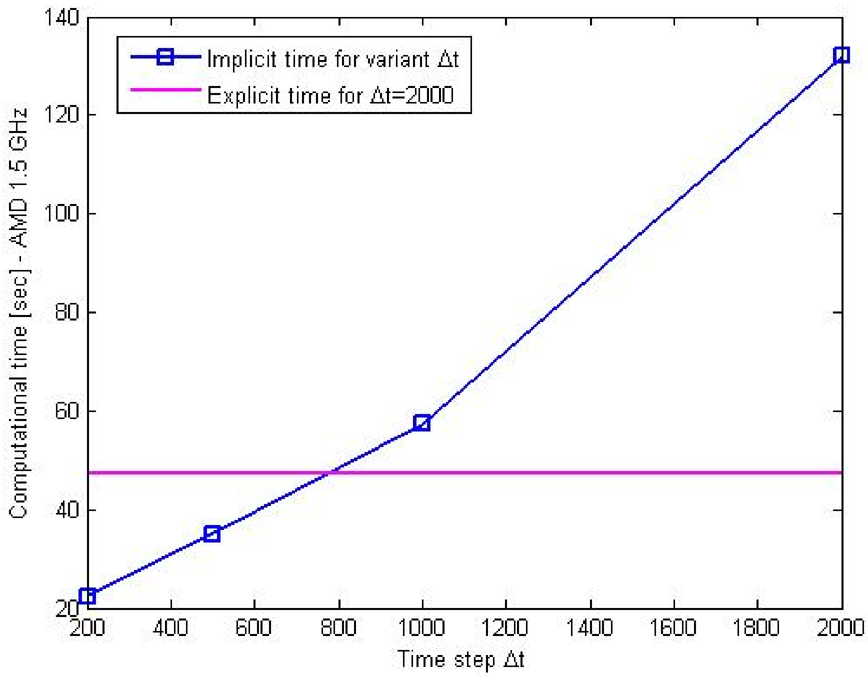


Figure 4.10.: Computational time (sec) for Implicit method for different time step size dt comparing to the explicit method time.

result from simulations in a laptop with AMD 1.5 GHZ processor and is 11.56 sec. Moreover, we simulate the implicit scheme initially with the same timr step dt and after with $2dt$, $10dt$,$100dt$.

Part IV.

Forward and Futures

5. Forward Price

Historically forwards and futures have been used in the energy market for physical delivery. The buyer of the physical contract provides a payment as defined by the fixed contract price and the seller delivers the commodity to a predetermined location at conculsion of the contract. If we consider the NYMEX natural gas markets we note market participants could enter into contracts to purchase natural gas deliveries in future months for fixed prices.

5.1. Contango and Backwardation

While the word contango may sound mysterious, it is used to describe a fairly normal pricing situation in futures. A market is said to be in contango when the forward price of a futures contract is above the expected future spot price $F_{t,T} > E_t[S_T] > S_t$. Contango occurs when speculators, to make a profit (on average), will short (sell) the forward contract with the expectation that as time passes the forward contract will decline to match the expected lower spot price. Normal backwardation, which is essentially the opposite of contango, occurs when the forward price of a futures contract is below the expected future spot price $F_{t,T} < E_t[S_T] < S_t$. Because contango and backwardation are known states in the market, traders can employ strategies that attempt to exploit them.

5.2. Forward Price PDE

Again we derive the differential equation for the forward price by applying arbitrage-free arguments to a risk-free portfolio formed by hedging a forward contract with stock shares [Mastro]. The basic geometric Brownian motion model of a stock is given by

$$dS_t = \mu S_t dt + \sigma S_t dW \tag{5.2.1}$$

where, again, μ is the mean rate return, σ is the annual volatility and $dW \sim N(0, 1)$. We will assume that the underlying randomness in the forward price and volatility is the same. This assumption allows the randomness in n shares of the underlying stock to be hedged with a forward contract. This creates a risk-free portfolio with a total value

$$\Pi_t = F_{t,T} + nS_t \tag{5.2.2}$$

Since changes in the asset (stock) price are linked to changes in the forward contract price, the change in portfolio value is written as

$$d\Pi_t = dF_{t,T} + ndS_t \tag{5.2.3}$$

A perfectly hedged portfolio will earn the risk-free rate r as given by

$$d\Pi_t = rnS_t dt = dF_{t,T} + ndS_t \tag{5.2.4}$$

Similar to arguments used previously opportunities would exist if a risk-free portfolio earned more or less than the risk-free rate.

Ito's lemma allows us to define the instantaneous price change of the forward price as

$$dF = \frac{\partial F}{\partial S} dS + \frac{\partial F}{\partial t} dt + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}(dS)^2 \tag{5.2.5}$$

Substituting (5.2.1) into (5.2.5) the forward differential spot price is

$$dF = \left(\frac{\partial F}{\partial S}\mu S_t + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\sigma^2 S_t^2\right) dt + \frac{\partial F}{\partial S}\sigma S_t dW \tag{5.2.6}$$

and inserting (5.2.6) into (5.2.4) gives

$$\begin{gathered}
rnS_t dt - ndS_t = dF_{t,T} = \left(\tfrac{\partial F}{\partial S}\mu S_t + \tfrac{\partial F}{\partial t} + \tfrac{1}{2}\tfrac{\partial^2 F}{\partial S^2}\sigma^2 S_t^2\right) dt + \tfrac{\partial F}{\partial S}\sigma S_t dW \\
\left(\tfrac{\partial F}{\partial S}\mu S_t + \tfrac{\partial F}{\partial t} + \tfrac{1}{2}\tfrac{\partial^2 F}{\partial S^2}\sigma^2 S_t^2 - rnS_t\right) dt + n\overbrace{(\mu S_t dt + \sigma S_t dW)}^{dS_t} + \tfrac{\partial F}{\partial S}\sigma S_t dW = 0 \\
\left(\tfrac{\partial F}{\partial S}\mu S_t + \tfrac{\partial F}{\partial t} + \tfrac{1}{2}\tfrac{\partial^2 F}{\partial S^2}\sigma^2 S_t^2 - rnS_t + n\mu S_t\right) dt + (\tfrac{\partial F}{\partial S}\sigma S_t + n\sigma S_t) dW = 0 \\
\left(\tfrac{\partial F}{\partial S}\mu S_t + \tfrac{\partial F}{\partial t} + \tfrac{1}{2}\tfrac{\partial^2 F}{\partial S^2}\sigma^2 S_t^2 - rnS_t + n\mu S_t\right) dt + \underbrace{(\frac{\partial F}{\partial S} + n)\sigma S_t dW}_{:=0} = 0
\end{gathered} \tag{5.2.7}$$

Since there cannot be portfolio risk associated with random movements dW, the portfolio is continuously hedged such that the number of shares is $n = -\frac{\partial F}{\partial S_t}$. The final differential equation for the forward price is

$$\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\sigma^2 S_t^2 + r\frac{\partial F}{\partial S_t} S_t = 0 \tag{5.2.8}$$

5.3. Convenience Yield

Commodity forward prices have an independent stochastic behavior not explained by spot prices. Since the convenience yield (CY) appears as a factor which cannot be observed directly, stochastic filtering has been proposed as a strategy of choice for its estimation from observed market prices. The convenience yield is the sum of additional benefits for owning a commodity in the integral time $[t, T]$ compared to accepting delivery of the commodity at time T via forward contract. If there is not sufficient storage capacity the commodity prices follow strong seasonality patterns.

However, storage of energy products is costly and sometimes practically impossible like in the case of electricity. So the convenience yield is defined as

$$\delta = \textbf{Benefit owning commodity} - \textbf{Cost of carry} \tag{5.3.1}$$

5.3.1. Forward Price with Fixed Convenience Yield

The convenience yield corrects for the added value of owning the connodity relative to owning a forward contract. The convenience yield term summarizes the prediction of market participants on the future supply of the commodity. An expected low supply translates into a premium for the owning the commodity, i.e. high convenience yield. In contrast, an expected abundant supply manifests as a low convenience yield.

So, the price process of the commodity from the nonholder's view is

$$dS_t^{nonholder} = (\mu - \delta)S_t dt + \sigma S_t dW, \tag{5.3.2}$$

and from the holder's view is

$$dS_t^{holder} = \mu S dt + \sigma S_t dW. \tag{5.3.3}$$

Rewritting the change in portfolio value at time t over a time period dt gives

$$d\Pi_t = dF_{t,T}^{nonholder} + n dS_t^{holder}, \tag{5.3.4}$$

which will still earn the risk-free rate r as described by

$$d\Pi_t = rnS_t^{holder} = dF_{t,T}^{nonholder} + n dS_t^{holder}. \tag{5.3.5}$$

In the same way used previously [Mastro] in the derivation of the forward price differential equation, the differential equation for the forward price with a fixed convenience yield δ can now be written as

$$\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\sigma^2 S_t^2 + (r-\delta)\frac{\partial F}{\partial S_t}S_t = 0 \quad (5.3.6)$$

where the drift must equal to the risk-free rate minus the convenience yield, else arbitrage opportunities would be present. Assuming the forward price is linear in S, the fair value is

$$F_0 = S_0 e^{(r-\delta)T} \quad (5.3.7)$$

5.3.2. Forward Price with Stochastic Convenience Yield

Adding a stochastic convenience yield

$$d\delta_t = k(a-\delta_t)S_t d_t + \sigma_2 dW_2 \quad (5.3.8)$$

naturally requires a derivation that includes the movement of the forwards price relative to the convenience yield. Again, we will form a portfolio composed of underlying commodity and an unspecified number of futures contracts.

Ito's lemma allows us to define the instantaneous price change of the forwards price as seen by a nonholder, taking into account the stochastic convenience yield

$$\begin{aligned} dF_{t,T}^{nonholder} &= \frac{\partial F_{t,T}}{\partial S}dS_t^{nonholder} + \frac{\partial F_{t,T}}{\partial t}dt + \frac{1}{2}\frac{\partial^2 F_{t,T}}{\partial S^2}(dS_{t,T}^{nonholder})^2 \\ &+ \frac{\partial F_{t,T}}{\partial \delta}d\delta_t + \frac{1}{2}\frac{\partial^2 F_{t,T}}{\partial \delta^2}(d\delta_{t,T})^2 + \frac{\partial F_{t,T}}{\partial S}\frac{\partial F_{t,T}}{\partial \delta}dS_t^{nonholder}d\delta_t \end{aligned}$$

Substituting the stochastic spot price model as seen by a nonholder and the stochastic convenience yield model gives

$$\begin{aligned} dF_{t,T}^{nonholder} &= \{\frac{\partial F}{\partial S}\left[(\mu-\delta_t)S_t\right] + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\sigma_1^2 S_t^2 \\ &+ \frac{\partial F}{\partial \delta}\left[k\left(a-\delta_t\right)\right] + \frac{1}{2}\frac{\partial^2 F}{\partial \delta^2}\sigma_2^2 + \frac{\partial^2 F}{\partial S \partial \delta}(S_t\sigma_1\sigma_2\rho)\}dt \\ &+ \frac{\partial F}{\partial S}(\sigma_1 S_t dW_1) + \frac{\partial F}{\partial \delta}(\sigma_2 dW_2) \end{aligned}$$

The change in the forwards and spot price is substituted into the change in portfolio value to give

$$\begin{aligned}&\{\tfrac{\partial F}{\partial S}\left[\left(\mu-\delta_t\right)S_t\right]+\tfrac{\partial F}{\partial t}+\tfrac{1}{2}\tfrac{\partial^2 F}{\partial S^2}\sigma_1^2 S_t^2\\&+\tfrac{\partial F}{\partial \delta}\left[k\left(a-\delta_t\right)\right]+\tfrac{1}{2}\tfrac{\partial^2 F}{\partial \delta^2}\sigma_2^2+\tfrac{\partial^2 F}{\partial S\partial\delta}(S_t\sigma_1\sigma_2\rho)\\&-rnS_t+n\mu S_t\}dt+\left(\underbrace{\frac{\partial F}{\partial S}+n}_{\frac{\partial F}{\partial S}=-n}\right)\sigma_1 S_t dW_1\\&+\tfrac{\partial F}{\partial \delta}\left(\sigma_2 dW_2\right)=0\end{aligned} \tag{5.3.9}$$

The stochastic convenience yield model predicts that the value of the futures in a frictionless market (without transaction costs) absent any risk-free arbitrage must follow the following partial differencial equation

$$\begin{aligned}\frac{\partial F}{\partial S}\left[\left(r-\delta_t\right)S_t\right]+\frac{\partial F}{\partial t}+\frac{1}{2}\frac{\partial^2 F}{\partial S^2}\sigma_1^2 S_t^2+\frac{\partial F}{\partial \delta}\left[k\left(a-\delta_t\right)+\sigma_2\lambda\right]\\+\frac{1}{2}\frac{\partial^2 F}{\partial \delta^2}\sigma_2^2+\frac{\partial^2 F}{\partial S\partial\delta}(S_t\sigma_1\sigma_2\rho)=0\end{aligned}$$

where λ is the market price of the convenience yield risk. The derivation was based on eliminating the risk of stochastic movements in the spot price; however, the investment is not riskless as the convenience yield risk cannot be hedged away.

The underlying assumption of no-arbitrage implies that the expected return beyond the risk-fee rate is related to the market price of risk. Risk arises from the stochastic behavior of the convenience yield [GS90], hence

$$\mu_F=r+\frac{\lambda' S\frac{\partial F}{\partial S}\sigma_1}{F}+\frac{\lambda\frac{\partial F}{\partial \delta}\sigma_2}{F} \tag{5.3.10}$$

where λ' is the market price per unit of price risk and λ is the market price per unit of convenience yield risk.

5.4. Schwartz model - Stochastic CY Model

In the 1990s, several mean reversion models were developed to describe the behavior of commodity prices. It well known that the convenience yield changes over time, but a major advantage was the introduction of a stochastic convenience yield [Schwartz97]. When supply is high, there is a little convenience or necessity to have the commodity on hand. When supply is low, the convenience yield will be high and the spot price will be higher than a long-dated futures contract.

The spot price of the commodity and the instantaneous convenience yield are assumed to follow the joint stochastic process:

$$\begin{aligned} dS_t &= (\mu - \delta_t)S_t dt + \sigma_1 S_t dW_1, \\ d\delta_t &= k(a - \delta_t)dt + \sigma_2 dW_2, \end{aligned} \tag{5.4.1}$$

with Brownian motions W_1 and W_2 under the objective measure $\mathbb{P}$ and correlation $dW_1 dW_2 = \rho dt$.

Under the risk-free pricing measure $\mathbb{Q}$ the dynamics are of the form

$$\begin{aligned} dS_t &= (r - \delta_t)S_t dt + \sigma_1 S_t d\tilde{W}_1, \\ d\delta_t &= [k(a - \delta_t) - \lambda]dt + \sigma_2 d\tilde{W}_2, \end{aligned} \tag{5.4.2}$$

where the constant λ denotes the market price of convenience yield risk and $\tilde{W}_1$ and $\tilde{W}_2$ are $\mathbb{Q}$-Brownian motions. It may be handy to introduce a new mean-level for the convenience yield process under $\mathbb{Q}$

$$\tilde{a} = a - \frac{\lambda}{k}, \tag{5.4.3}$$

which leads to the dynamics

$$d\delta_t = k(\tilde{a} - \delta_t)dt + \sigma_2 d\tilde{W}_2. \tag{5.4.4}$$

5.4.1. Joint Distribution of State Variables

The log-spot $X_t = log(S_t)$ and the convenience yield δ_t are jointly normally distributed. The transition density is

$$\begin{pmatrix} X_t \\ \delta_t \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_X(t) \\ \mu_\delta(t) \end{pmatrix}, \begin{pmatrix} \sigma_X^2(t) & \sigma_{X\delta}(t) \\ \sigma_{X\delta}(t) & \sigma_\delta^2(t) \end{pmatrix}\right) \tag{5.4.5}$$

with parameters

$$\begin{aligned} \mu_X(t) &= X_0 + (\mu - \tfrac{1}{2}\sigma_1^2 - a)t + (a - \delta_0)\tfrac{1-e^{-kt}}{k} \\ &\mu_{\delta(t) = e^{-kt}\delta_0 + a(1-e^{-kt})} \\ \sigma_X^2(t) &= \tfrac{\sigma_2^2}{k^2}(\tfrac{1}{2k}(1 - e^{-2kt}) - \tfrac{2}{k}(1 - e^{-kt}) + t) + 2\tfrac{\sigma_1\sigma_2\rho}{k}(\tfrac{1-e^{-kt}}{k} - t) + \sigma_1^2 t \\ \sigma_\delta^2(t) &= \tfrac{\sigma_2^2}{2k}(1 - e^{-2kt}) \\ \sigma_{X\delta}(t) &= \tfrac{1}{k}\{(\sigma_1\sigma_2\rho - \tfrac{\sigma_2^2}{k})(1 - e^{-kt}) + \tfrac{\sigma_2^2}{2k}(1 - e^{-2kt})\} \end{aligned} \tag{5.4.6}$$

The mean-parameters given in (5.4.6) refer to the $\mathbb{P}$-dynamics. To obtain the parameters under $\mathbb{Q}$ one can simply replace μ by r and a by $\tilde{a}$ defined in equation (5.4.3). Let the $\mathbb{Q}$-parameters be denoted by $\tilde{\mu}_X(t)$ and $\tilde{\mu}_\delta(t)$.

5.4.2. Futures Price

It is worth to mention that the futures and forward price coincide since in our model the interest rate is assumed to be constant. Let the futures price at time t with time to maturity $\tau = T - t$ be $F(S_t, \delta_t, t, T)$.

At time zero the futures price is given by the $\mathbb{Q}$-expectation of S_T.

$$F(S_t, \delta_t, 0, T) = E^{\mathbb{Q}}(S_T) = exp\{\mu_X(T) + \frac{1}{2}\sigma_X^2(T)\} = S_0 e^{A(T)-B(T)\delta_0} \tag{5.4.7}$$

with

$$A(T) = (r - \tilde{a} + \frac{1}{2}\frac{\sigma_2^2}{k^2} - \frac{\sigma_1\sigma_2\rho}{k})T + \frac{1}{4}\sigma_2^2\frac{1-e^{-2kT}}{k^3}$$
$$+(k\tilde{a} + \sigma_1\sigma_2\rho - \frac{\sigma_2^2}{k})\frac{1-e^{-kT}}{k^2}$$

$$B(T) = \frac{1-e^{-kT}}{k} \tag{5.4.8}$$

α	σ_1	σ_2	r	κ	λ	ρ
0.0699	0.3630	0.4028	0.0373	1.4221	-0.0183	0.8

Table 5.1.: Calibrated parameters set for WTI [Schwartz97].

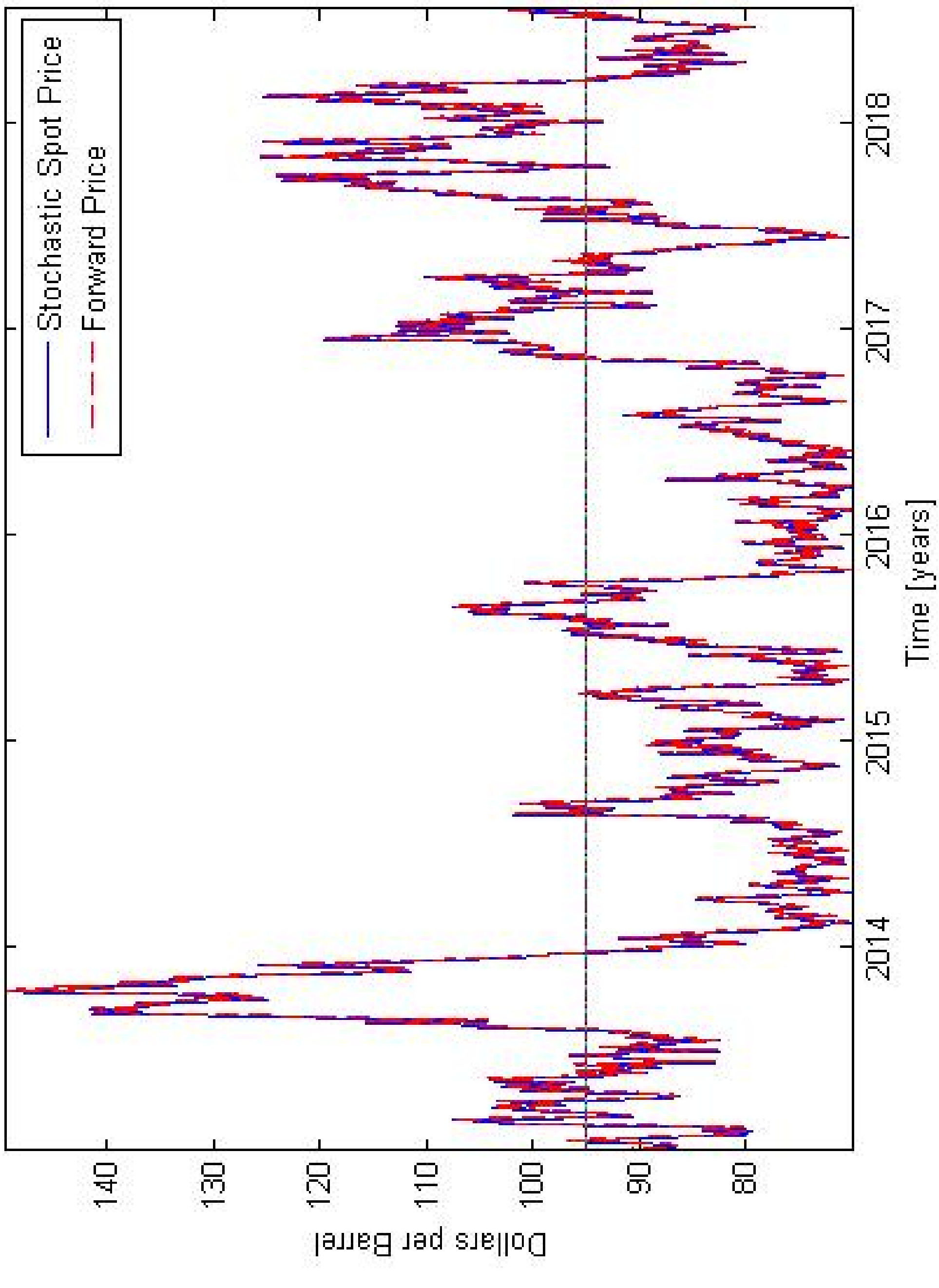


Figure 5.1.: Simulation of the stochastic spot and futures price for the next 5 years from the parameter set of Table 5.1.

6. Futures Options

It is more natural to trade options on commodity futures rather than options on the actual underlying asset. Why would anyone write an option on futures, instead of writing it on the cash instrument directly?

In fact, the advantages of such contracts are many, and the fact that option contracts written on futures are the most liquid is not a coincidence. First of all, if one were to buy and sell the underlying in order to hedge the option positions, the futures contracts are more convenient. They are more liquid, and they do not require upfront cash payments. Second, hedging with cash instruments could imply, for example, selling or buying thousands of barrels of oil.Where would a trader put so much oil, and where would he get it? Worse, dynamic hedging requires adjusting such positions continuously. It would be very inconvenient to buy and sell a cash underlying. Long and short positions in futures do not result in delivery until the expiration date. Hence, the trader can constantly adjust his or her position without having to store barrels of oil at each rebalancing of the hedge. Futures are also more liquid and the associated transactions costs and counterpart risks are much smaller.

Most futures options are American style. Exercising a futures call option gives the holder possession of the future contract as well as a marked to market cash payment for the difference between the strike price K and the most recent (e.g. last trading day's) futures settlement price $F_{t-1} - K$. Subsequently, the owner can hold the futures contract or sell the futures contract for the current futures price for a net profit of $F_t - F_{t-1}$

6.1. Futures Risk & Neutral Behavior

The main source of risk in a futures contract is the basis risk. The exchange requires daily settlement of the futures contract in margin account. When you open a futures contract, the futures exchange will state a minimum amount of money that you must deposit into your account. This original deposit of money is called the initial margin. When your contract is liquidated, you will be refunded the initial margin plus or minus any gains or losses that occur over the span of the futures contract. In other words, the amount in your margin account changes daily as the market fluctuates in relation to your futures contract. The minimum-level margin is determined by the futures exchange and is usually 5% to 10% of the futures contract.

In the risk-neutral world, a future behaves similar to a stock with zero drift paying a dividend at a rate r. The initial entry into futures contract requires no payment. This mean that the contract has zero value at time zero. At the end of the first day, the contract is marked to market for a positive or negative payoff of $F_1 - F_0$. Discounting the daily-denominated risk free rate gives the risk-neutral value of the payoff at time zero as

$$e^{-r\Delta t}E_0^Q[F_1 - F_0] = 0 \tag{6.1.1}$$

This can be repeated over several time periods to show that the expected drift is zero

$$F_0 = E_0^Q[F_1] = \ldots = E_0^Q[F_T] \tag{6.1.2}$$

in the risk-neutral world where the money market account is the numerate.

6.2. Futures Contract For Constant Interest Rate

The interest rate r is assumed to be normalized to a rate per day. The initial is zero. At the close of trading on the first day, the mark to market requirement from a future exchange created a profit (or loss) of $F_1 - F_0$, which is invested (or borrowed) at the risk free for the remaining $n-1$ days

$$\underbrace{e^{r(1)}}_{\#contracts}\ \underbrace{(F_1 - F_0)}_{mark\,to\,market}\ \overbrace{e^{r(n-1)}}^{invest\,n-1\,days} = (F_1 - F_0)e^{r(n)} \tag{6.2.1}$$

Furthermore, at the end of the first day, e^r more contracts are entered to give a new total number of contracts of $e^r + e^{r(1)} = e^{r(2)}$. At the end of the second trading day, the mark to market creates a new profit (or loss) of $F_2 - F_1$ which is invested (or borrowed) at the risk free for the remaining $n-2$ days

$$\underbrace{e^{r(2)}}_{\#contracts}\ \underbrace{(F_2 - F_1)}_{mark\,to\,market}\ \overbrace{e^{r(n-2)}}^{invest\,n-2\,days} = (F_2 - F_1)e^{r(n)} \tag{6.2.2}$$

Again, at the end of the first day, e^r more contracts are entered to give a new total number of contracts of $e^r + e^{r(2)} = e^{r(3)}$. Similar streams of investments or loan are made for each day up to last trading day n. On this last trading day the value is found by adding all streams

$$\sum_{i=1}^{n} \underbrace{e^{r(2)}}_{\#contracts} \underbrace{(F_2 - F_1)}_{mark\,to\,market} \overbrace{e^{r(n-2)}}^{invest\,n-i\,days} = \sum_{i=1}^{n}(F_i - F_{i-1})e^{r(n)} = (F_n - F_0)e^{r(n)} \quad (6.2.3)$$

where the closing futures price must converge to the closing spot price, $F_n = S_T$. This assumption is called *convergence assumption*.

6.2.1. Put-Call Parity: Hedging With Futures

An important principle in options pricing is called a put-call parity. It says that the value of a call option, at one strike price, implies a certain fair value for the corresponding put, and vice versa. This relationship is expressed in the formula

$$V_c + Ke^{-rt} = V_p + F_0 \quad (6.2.4)$$

If the parity is violated, an opportunity for arbitrage exists. Our goal is to show that the value at time t of the fixed-price forward contract for a party that at delivery pays a fixed price K and receives one unit of the commodity is

$$V_c(t, F_t) = e^{-r(T-t)}(F_t - K) \quad (6.2.5)$$

where r is the risk-free rate.

Alternatively, the value of this contract for a party that delivers one unit of the commodity in exchange for a fixed payment is

$$V_p(t, F_t) = e^{-r(T-t)}(K - F_t) \quad (6.2.6)$$

To prove these two equalities we use the delta hedging procedure with hedges applied at certain times $t_0, t_1, \ldots, t_n = T$. Successful delta hedging is equivalent to the following equality

$$V(T, F_T) + e^{r(T-t_1)}\delta_0 \Delta F_0 + e^{r(T-t_2)}\delta_1 \Delta F_1 + \ldots + e^{r(T-t_{i+1})}\delta_i \Delta F_i = e^{r(T-t_0)}V(t_0, F_{t_0})$$

where

$$\delta_i = \frac{\partial V}{\partial F}(t_i, F_{t_i}) \; and \; \Delta F_i = F_{t_{i+1}} - F_{t_i} \quad (6.2.7)$$

6.2.2. Delta Hedging

The previous equality is the essence of dynamic or delta hedging with futures: the asset value at the beginning of hedging period is equal to the asset value plus the value of the hedges at the end of the period (all adjusted for the accrued interest). In other words, delta hedging preserves the value of the asset by ensuring that the value of the portfolio consisting of the asset and the hedges does not change with time (This is a somewhat idealized picture since we have ignored the transaction costs associated with hedging, which in general should not be).

Assume now for example that a gas vendor A wants to trade a future contract but does not agree with the payer's future contract evaluation V_c (6.2.5), or with the receiver's contract value V_p (6.2.6).

Case 1 A is confident that the value at t_0 of the receiver's future contract (A receives a fixed payment) is greater than the one given by V_p,

$$V_p^A(t_0, F_{t_0}) > e^{-r(T-t_0)}(K - F_{t_0}). \tag{6.2.8}$$

So vendor A is willing to pay today V_p^A for the right to deliver a commodity in the future month and to receive a fixed payment K. After entering into the contract with A, we immediately implement the delta hedging strategy, assuming that V_p is the correct value of the contract (from our point of view, it is a payer's contract). At expiration time T we pay A the fixed payment K, and receive the commodity that we promptly sell at the spot market for S_T. Thus, at expiration our total cash flow per unit of commodity consists of the following components:

- $+S_T$ is the sale of the commodity at the spot market.
- $-K$ is the fixed payment.
- $e^{r(T-t_0)}\delta_0\Delta F_0 + \ldots + \delta_{n-1}\Delta F_{n-1}$ is the futures margin account.
- $+e^{r(T-t_0)}V_p^A(t_0, F_{t_0})$ is the initial payment from A at the inception of the contract.

Our total P&L (Profit & Loss) from the contract is

$$\Pi = (S_T - K) + e^{r(T-t_0)}\delta_0\Delta F_0 + \ldots + \delta_{n-1}\Delta F_{n-1} + e^{r(T-t_0)}V_p^A(t_0, F_{t_0})$$

$$\Pi = e^{r(T-t_0)}\left[\underbrace{V(t_0, F_{t_0}) + e^{-r(T-t_0)}\left(F_{t_0} - K\right)}_{>0}\right] > 0.$$

Therefore, at expiration we guarantee ourselves a profit regardless of market behavior. We just borrow the necessary cash and repay the debt at expiration. This is an arbitrage opportunity and a common assumption is that arbitrage cannot exist for a long period of time. Indeed, once A will realize that he has overvalued the contract and allows us to make riskless profit, he will adjust the value.

Case 2 Assume now that the gas vendor A believes that

$$V_p^A(t_0, F_{t_0}) < e^{-r(T-t_0)}(K - F_{t_0}). \tag{6.2.9}$$

Thus, if we are interested in entering into a receiver's contract with A (we receive a fixed payment), we will be happy to learn that A values this contract lower than we do. Therefore, A will accept the payment of $V_p^A(t_0, F_{t_0})$ for agreeing to receive the commodity in the specified future month in exchange for paying the fixed payment K. Here of course we can implement delta hedging, assuming that V_p holds, obtaining again that $\Pi > 0$. This is again an arbitrage opportunity.

6.3. Blacks's Model For Future Options Pricing

In Black's model [Black76], the future price is log-normally distributed and based on the stochastic process

$$dF = \sigma F dW, \tag{6.3.1}$$

where σ is the annualized volatility and dW a geometric Brownian motion. The risk-neutral distribution at expiration of the contracts equivalent to a stock that pays a dividend yield equal to the risk-free interest rate

$$ln(F_T) \sim \phi(lnF_0 - \frac{1}{2}\sigma^2, \sigma^2 T). \tag{6.3.2}$$

The probability of finding a random variable x, drawn from a Gaussian distribution of mean μ and variance σ^2given by the probability density function (PDF) as

$$\phi(x, \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{1}{\sigma}\phi^n\left(\frac{x-\mu}{\sigma}\right) \tag{6.3.3}$$

where $\phi^n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$is the normalized PDF.

The standard deviation per time step (e.g per day) is $\sigma_{sd} = \sigma\sqrt{T}$. A normal probability distribution with a mean of zero ($\mu = 0$) and unity standard deviation, $N(0, 1)$, is achieved by employing a change of variables

$$Q = \frac{lnV - \mu}{\sigma_{sd}}. \tag{6.3.4}$$

Then the normalized probability is given by

$$\phi^n(Q) = \frac{1}{\sqrt{2\pi}} e^{-\frac{Q^2}{2}} \tag{6.3.5}$$

At time T, the payout from the call option is

$$E\left[max(V_T - K, 0)\right] = \int_K^\infty (V - K)\phi_V(V)dV \tag{6.3.6}$$

Changing the limits of integration under the density function for Q gives

$$E\left[max(V_T - K, 0)\right] = \int_Q^\infty (e^{Q\omega + \sigma_{sd}} - K)\phi_Q(Q)dQ \tag{6.3.7}$$

Solving the first term in the integral gives

$$\begin{aligned}(e^{Q\omega+\sigma_{sd}} - K)\phi_Q(Q) = (e^{Q\omega+\sigma_{sd}})\frac{1}{\sqrt{2\pi}}e^{-\frac{Q^2}{2}} = \frac{1}{\sqrt{2\pi}}e^{\frac{2Q\sigma_{sd}+2\sigma_{sd}-Q^2}{2}} \\ = \frac{1}{\sqrt{2\pi}}e^{\frac{-(Q-\sigma_{sd})^2+2\mu+\sigma_{sd}^2}{2}} = e^{\mu+\sigma_{sd}^2/2}\phi_{Norm}(Q-\sigma_{sd})\end{aligned}$$

Substituting into the expectation gives

$$\begin{aligned}E\left[max(V_T - K, 0)\right] = \int_Q^\infty (e^{Q\omega+\sigma_{sd}})\phi_Q(Q)dQ - \int_Q^\infty K\phi(Q)dQ \\ = e^{\mu+\sigma_{sd}^2/2}\int_Q^\infty \phi(Q-\sigma_{sd})dQ - \int_Q^\infty K\phi(Q)dQ\end{aligned}$$

The integrals are summing the probability of finding a variable from the lower limit to infinity. Conveniently, the probability $\Phi^{(n)}$ of finding a random variable in the interval $(-\infty, x)$ drawn from the standard Gaussian distribution ($z \sim N(0,1)$) is the normalized cumulative distribution function (CDF) which is given by

$$\Phi^{(n)}(x) = \int_{-\infty}^x \phi^n(s)ds = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}s^2 ds} = \frac{1}{2}\left[1 + erf\left(\frac{x}{\sqrt{2}}\right)\right] \tag{6.3.8}$$

where erf is the error function.

$$1 - \int_{Q}^{\infty} \phi(Q - \sigma_{sd})dQ = 1 - \Phi^{(n)} \left[\left(\frac{\overbrace{lnK - \mu}^{Q}}{\sigma_{sd}} \right) - \sigma_{sd} \right]$$
$$= \Phi^{(n)} \left[\left(\frac{-lnK + lnE(V) - \frac{\sigma_{sd}^2}{2}}{\sigma_{sd}} \right) + \sigma_{sd} \right]$$
$$= \Phi^{(n)} \left[\overbrace{\frac{ln(E(V)/K) + \frac{\sigma_{sd}^2}{2}}{\sigma_{sd}}}^{d1} \right]$$

the second integral is given by

$$1 - \int_{Q}^{\infty} \phi(Q)dQ = \Phi^{(n)} \left[\left(\frac{-lnK + lnE(V) - \frac{\sigma_{sd}^2}{2}}{\sigma_{sd}} \right) \right]$$
$$= \Phi^{(n)} \left[\overbrace{\frac{ln(E(V)/K) - \frac{\sigma_{sd}^2}{2}}{\sigma_{sd}}}^{d2} \right]$$

From the previous we can take

$$c = e^{-rT} \left[F_0 \Phi^{(n)} \left[\overbrace{\frac{ln(E(V)/K) + \frac{\sigma_{sd}^2}{2}}{\sigma_{sd}}}^{d1} \right] - K\Phi^{(n)} \left[\overbrace{\frac{ln(E(V)/K) - \frac{\sigma_{sd}^2}{2}}{\sigma_{sd}}}^{d2} \right] \right] \tag{6.3.9}$$

More compactly, the call and the put options on futures contracts are written as

$$\begin{aligned} c &= e^{-r(T-t)}[F_0\Phi^{(n)}(d1) - K\Phi^{(n)}(d2)] \\ p &= e^{-r(T-t)}[K\Phi^{(n)}(-d2) - F_0\Phi^{(n)}(-d1)] \end{aligned} \tag{6.3.10}$$

where t accounts for a non-zero present time. The values of call options on futures contracts calculated from Black's model is shown in Figure 6.1.

The Greeks calculate the sensitivity of the derivative price (or portfolio of derivatives). Delta is the rate of change of the option price with respect to the underlying

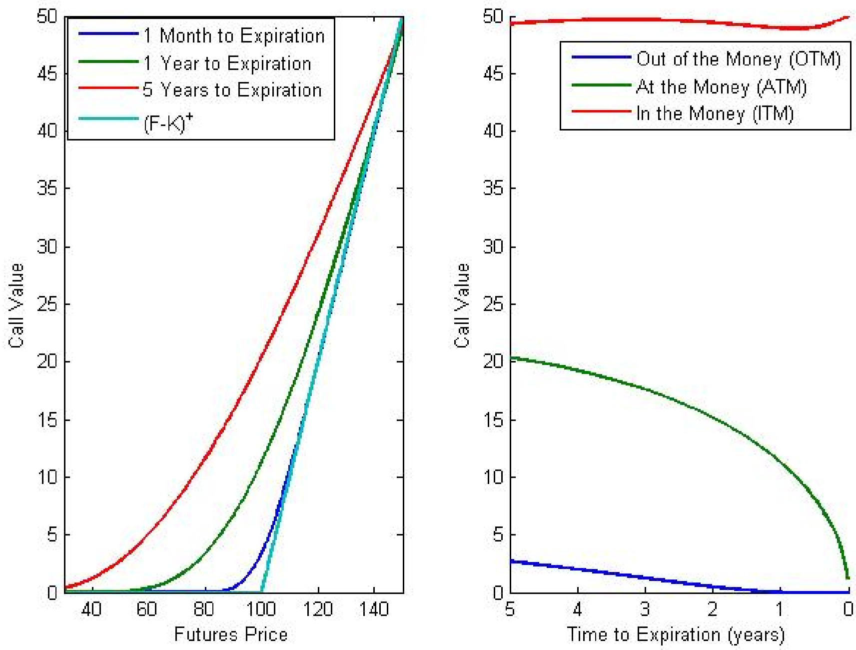


Figure 6.1.: European call futures option value as a function of futures price (left) and time to expiration (right). An OTM option has a time value based on the probability that the option will finish ITM.

asset. A riskless portfolio is formed with delta shares of the futures contract with one short on a call option on the futures contract. The gain (loss) from the shorted call option would be offset by a corresponding loss (gain) in futures contract. This delta neutrality can only be maintained in a narrow window of futures price, time to maturity, interest rates, and certain price or time movement, the delta will have changed by a sufficient amount to require portfolio rebalancing.

6.4. American Options

American options based on stocks as well as futures are the most frequently traded options in the market. The option for early exercise prevents the development of a closed form solution. Popular techniques to price American style options are numerical approaches based on finite difference method as we showed in the previous chapter. An interesting alternative is the Barone-Adesi and Whaley (BAW) model. BAW model is a quadratic approximation method for pricing exchange-traded American call and put options on commodities and commodity futures and is based on the Black-Scholes model. Finite-difference methods provide similar results but are more difficult and expensive to use. The BAW model has the advantages of being fast, accurate and inexpensive to use. Also, it is most accurate for options that will

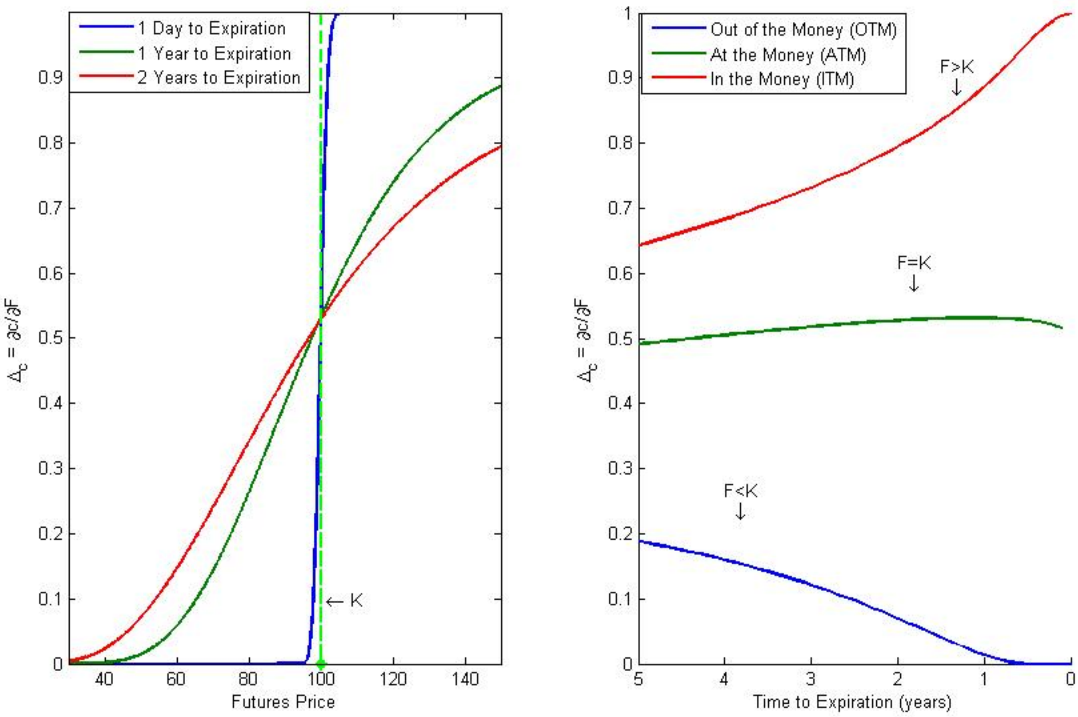


Figure 6.2.: The dependence of Delta on futures price (left) and on the time to expiration (right) within Black's model

expire in less than one year.

BAW provides an approximation to stocks and commodities with a continuous cost of carry b, composed of a continuous dividend yield d, convenience yield y, and storage u, and given by

$$b = r - d - y + u. \tag{6.4.1}$$

Usually a stock pays a dividend yield of zero (non-dividend paying) or less than the risk free rate, $d < r$; thus, $b = r - d > 0$. A futures contract has a cost of carry $b = 0$. As the cost of carry of a stock is usually higher than the cost of carry (equal to zero) of a futures contract, an option on future contract for a stock typically will be worth more than an option on the same stock. When $b \geq r$, An American call option is never exercised early and thus the American call price equals to the European call price.

A similar trend applies for commodities in which the future call options is higher than the call option as long as the convenience yield is less than the interest rate, $y < r$ thus $b = r - y > 0$. When the stock dividend rate exceeds the risk-free rate, $d > r \Rightarrow b = r - d < 0$, then the futures call option will be worth less than the call than the call option. Likewise for a commodity with a large convenience yield, the call option on the commodity price will exceed the price of the call option on the futures.

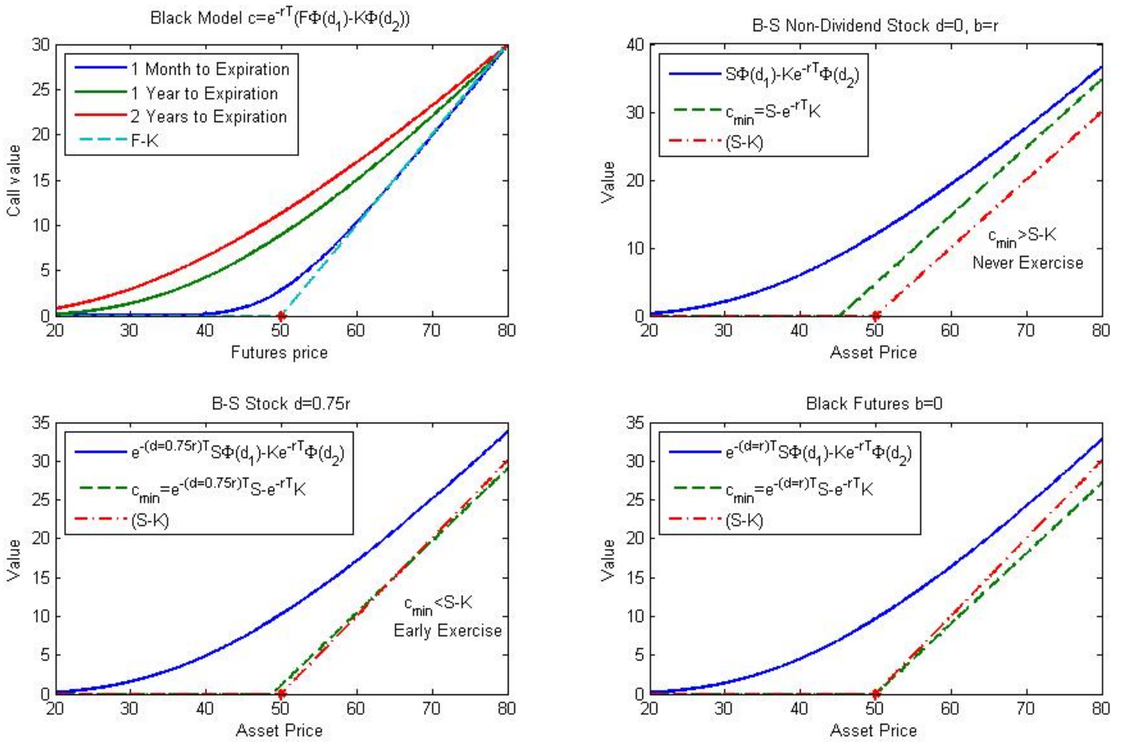


Figure 6.3.: (Up-left) European call futures option value as a function of futures price. Black-Scholes price for a call, Black-Scholes minimum price c_{min}, and the $S - K$ immediate profit lines (up-right) for non-dividend payment, (bottom-left) for $0 < b < r$ and (bottom right) zero cost of carry - futures option.

From the above figure the $b < r$, graph shows that the minimum European price is less than the early exercise $(S - K)$. Thus, early exercise is possible when the cost of carry is less than the risk-free rate.

6.4.1. American Option Derivation

An asset or a commodity S is described by the SDE

$$\frac{dS}{S} = \alpha dt + \sigma dW \qquad (6.4.2)$$

where α is the expected relative spot price change. Barone-Adesi and Whaley [BAW87] assume that a continuous stream of dividend payments d is made on S. The relationship of the asset or the underlying commodity to the futures price F is

$$F = Se^{bT} = Se^{(r-d-y+u)T} \tag{6.4.3}$$

The SDE for the future price is then

$$\frac{dF}{F} = (a-b)dt + \sigma dz \tag{6.4.4}$$

A portfolio formed by a riskless hedge of a contingent claim V and the stock or sport price of the underlying commodity gives the PDE

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + bS\frac{\partial V}{\partial S} - rV + \frac{\partial V}{\partial t} = 0 \tag{6.4.5}$$

The movement of the option price V is applicable to an American call, American put, or a European put.

An option on a futures contract can be priced in the same framework. The most important detail is that the cost to enter into a futures contract is zero and the cost of carrying a future contract is also zero. Therefore, $b = 0$ or equivalently the dividend is $d = r$. A portfolio formed by a riskless hedge of an option V and the futures contract gives the following PDE

$$\frac{1}{2}\sigma^2 F^2 \frac{\partial^2 V}{\partial S^2} - rV + \frac{\partial V}{\partial t} = 0 \tag{6.4.6}$$

with terminal conditions $c = (max[0, S_T - K])$ for a call and $p = (max[0, K - S_T])$ for a put yields the BS equations for a European option

$$\begin{gathered} c = e^{-rT}[Se^{bT}\Phi^{(n)}(d1) - K\Phi^{(n)}(d2)] = Se^{(b-r)T}\Phi^{(n)}(d1) - Ke^{-rT}\Phi^{(n)}(d2) \\ p = e^{-rT}[-Se^{bT}\Phi^{(n)}(-d1) + K\Phi^{(n)}(-d2)] = -Se^{(b-r)T}\Phi^{(n)}(-d1) + Ke^{-rT}\Phi^{(n)}(d2) \\ d1 = \frac{ln(Se^{bT}/K)+\frac{\sigma^2 T}{2}}{\sigma\sqrt{T}} \qquad d2 = \frac{ln(Se^{bT}/K)-\frac{\sigma^2 T}{2}}{\sigma\sqrt{T}} \end{gathered} \tag{6.4.7}$$

Similarly the terminal conditions $c = (max[0, F_T - K]$ and $p = (max[0, K - F_T])$ produce Black's model for futures options (where b=0)

$$\begin{gathered} c = e^{-rT}[F_0\Phi^{(n)}(d1) - K\Phi^{(n)}(d2)] \\ p = e^{-rT}[-F_0\Phi^{(n)}(-d1) + K\Phi^{(n)}(-d2)] \\ d1 = \frac{ln(F_0/K)+\frac{\sigma^2(T-t)}{2}}{\sigma\sqrt{(T-t)}} \qquad d2 = \frac{ln(F_0/K)-\frac{\sigma^2(T-t)}{2}}{\sigma\sqrt{(T-t)}} \end{gathered} \tag{6.4.8}$$

6.5. Optimal Exercise

It is well known that the main analytical problem in pricing American options is the calculation of the optimal exercise boundary . The exercise boundary is the critical price that separates the continuation region and the exercise region. More specifically:

$$\begin{array}{l} S > S^*(t) \rightarrow Exercise \\ S \leq S^*(t) \rightarrow Continue \end{array} \tag{6.5.1}$$

The critical price is a function of time, since the opportunity cost of exercising changes the closer we come to expiration. The characteristics of the optimal early exercise policies of American options depend critically on whether the underlying asset is non-dividend paying or dividend paying (discrete or continuous). We show that the optimal exercise boundary of an American call, with continuous dividend yield or zero dividend, is a continuous decreasing function of time of expiry τ

6.5.1. The Barone-Adesi and Whaley Model

American options gives the opportunity for early exercise. This opportunity adds a premium over the price of a European futures option and described by

$$\epsilon = C_{Am}(S,T) - c_{Eu}(S,T). \tag{6.5.2}$$

This additional premium ϵ will satisfy the same PDE as the American and European options satisfy

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \epsilon}{\partial S^2} + bS\frac{\partial \epsilon}{\partial S} - r\epsilon + \frac{\partial \epsilon}{\partial t} = 0. \tag{6.5.3}$$

Multiplying the previous PDE (6.5.3) by $2/\sigma^2$ and noting $\frac{\partial \epsilon}{\partial \tau} = -\frac{\partial \epsilon}{\partial t}$ [1]gives

$$S^2 \frac{\partial^2 \epsilon}{\partial S^2} + \frac{2b}{\sigma^2} S \frac{\partial \epsilon}{\partial S} - \frac{2r}{\sigma^2}\epsilon - \frac{2}{\sigma^2}\frac{\partial \epsilon}{\partial \tau} = 0. \tag{6.5.4}$$

Grouping constant factors as $M = 2r/\sigma^2$ and $N = 2b/\sigma^2$ gives

$$S^2 \frac{\partial^2 \epsilon}{\partial S^2} + NS\frac{\partial \epsilon}{\partial S} - M\epsilon - \frac{M}{r}\frac{\partial \epsilon}{\partial \tau} = 0. \tag{6.5.5}$$

Baroni-Adesi and Whaley divide the pricing premium ϵ_C of an American call option into a function of T and function of S and h (time decay)

[1]Barone-Adesi and Whaley define $\tau = T - t$ to evolve from the option expiration to present.

$$\epsilon_C = h(\tau)f(S,h), \tag{6.5.6}$$

with the corresponding derivatives

$$\frac{\partial^2 \epsilon}{\partial S^2} = h\frac{\partial^2 f}{\partial S^2}\frac{\partial \epsilon}{\partial S} = h\frac{\partial f}{\partial S}\frac{\partial \epsilon}{\partial \tau} = \frac{\partial h}{\partial \tau}f + h\frac{\partial h}{\partial \tau}\frac{\partial f}{\partial h}. \tag{6.5.7}$$

Substituting into the PDE (6.5.5) for the pricing premium and dividing by h

$$S^2\frac{\partial^2 f}{\partial S^2} + NS\frac{\partial f}{\partial S} - Mf - \frac{M}{r}\frac{\partial h}{\partial \tau}\frac{f}{h} - \frac{M}{r}\frac{\partial h}{\partial \tau}\frac{\partial f}{\partial h} = 0. \tag{6.5.8}$$

Defining the time decay as

$$h = 1 - e^{-r\tau} \qquad \frac{\partial h}{\partial \tau} = re^{-r\tau} = r(1-h), \tag{6.5.9}$$

yields

$$S^2\frac{\partial^2 f}{\partial S^2} + NS\frac{\partial f}{\partial S} - Mf - \frac{M}{r}r(1-h)\frac{f}{h} - \frac{M}{r}r(1-h)\frac{\partial f}{\partial h} = 0 \Rightarrow$$
$$S^2\frac{\partial^2 f}{\partial S^2} + NS\frac{\partial f}{\partial S} - \frac{M}{h}f - (1-h)\frac{\partial f}{\partial h} = 0.$$

The BAW approximation is to assume that the last term is approximately zero for two reasons: $\frac{\partial f(\tau \approx 0)}{\partial h} \approx 0$ for an option near expiration, and $h(\tau \approx 0) \approx 1$ for a long-dated option [2].

So the BAW approximate the early exercise premium differential equation as

$$S^2\frac{\partial^2 f}{\partial S^2} + NS\frac{\partial f}{\partial S} - \frac{M}{h}f = 0, \tag{6.5.10}$$

which is a second order ODE with two linearly independent solutions of the form aS^q. They can be found by substituting $f = aS^q$ into (6.5.9)

$$S^2 aq(q-1)S^{q-2} + NSaqS^{q-1} - \frac{M}{h}aS^q = 0 \tag{6.5.11}$$

$$aS^q[q^2 + (N-1)q - \frac{M}{h}] = 0 \tag{6.5.12}$$

[2] Ju and Zhong [JuZh99] continue the derivation with the last term to produce a numerical solution with better accuracy for intermediate maturity options.

The roots of (6.5.12) are

$$q_1 = \frac{-(N-1)-\sqrt{(N-1)^2+4\frac{M}{h}}}{2}, \quad q_2 = \frac{-(N-1)+\sqrt{(N-1)^2+4\frac{M}{h}}}{2}, \tag{6.5.13}$$

where $M/h > 0$, thus $q_1 < 0$ and $q_2 > 0$.

The general solution to the early exercise premium differential equation is

$$f = a_1 S^{q_1} + a_2 S^{q_2} \tag{6.5.14}$$

With q_1 and q_2 known, a_1 and a_2 to be determined. Examining the first term, $a_1 S^{q_1}$, where $q_1 < 0$, shows that an asset price near zero, $S \approx 0$, implies a very large premium, $f = S^{q_1} \approx \infty$. This is unacceptable since the early exercise premium of the American call becomes worthless when commodity price drops to zero. Therefore $a_1 = 0$ and this gives an intermediate expression for the American call option as

$$C_{Am} = c_{Eu} + h a_2 S^{q_2}, \tag{6.5.15}$$

As $S = 0$, $C_{Am}(S,T) = 0$ since both $c_{Eu}(S,T)$ and $h a_2 S^{q_2}$ are equal to zero. As S rises, the value of $C_{Am}(S,T)$ rises for two reasons: $c_{Eu}(S,T)$ rises and $h a_2 S^{q_2}$ rises, assuming $a_2 > 0$. In order to represent the value of the American call, however, the function on the right hand side of [9.45] should touch, but not intersect, the boundary imposed by the early exercise proceeds of the American call, $S - K$.

Above a critical asset price S^*, the value of the American call option will match the early exercise proceeds $C_{Am} = S - K$. At the critical asset price, the investor is indifferent to exercising the option or holding onto the option. The early exercise plus the European call should equal to the early exercise proceeds to give the first constraint

$$S^* - K = c_{Eu}(S^*, \tau) + h a_2 S^{*q_2}. \tag{6.5.16}$$

A second constraint is to equate the slope of the two lines

$$1 = e^{(b-r)T}\Phi[d_1(S^*)] + h q_2 a_2 S^{*q_2-1}, \tag{6.5.17}$$

where the early exercise proceeds has a slope of one and the slope of the European call is the Greek delta. There are two equations with two unknowns, a_2 and S^*. The slope equality can be solved for a_2 as

$$a_2 = \frac{1 - e^{(b-r)T}\Phi[d_1(S^*)]}{h q_2 S^{*q_2-1}}, \tag{6.5.18}$$

which can be substituted into the first constraint to give a master equation for the critical asset or commodity price

$$S^* - K = c_{Eu}(S^*, \tau) + \frac{1 - e^{(b-r)T}\Phi[d_1(S^*)]}{q_2} S^* \tag{6.5.19}$$

Although S^* is the only unknown value in equation (6.5.19), it must be determined iteratively in a computer simulation. With S^* known, equation (6.5.16) provides the value of a_2. Substituting (6.5.18) into (6.5.15) and simplifying yields

$$\begin{aligned} C_{Am}(S,\tau) = c_{Eu}(S,\tau) + A_2(S/S^*)^{q_2}, \qquad S < S^*, \\ C_{Am}(S,\tau) = S - K, \qquad S \geqslant S^*, \end{aligned} \tag{6.5.20}$$

where

$$A_2 = (\frac{S^*}{q_2})(1 - e^{(b-r)T}\Phi[d_1(S^*)]) \tag{6.5.21}$$

Note that $A_2 > 0$ since S^* are positive when $b < r$. Equation (6.5.20) is therefore an efficient analytic approximation of the value of an American call option written on commodity when the cost of carry is less than riskless rate of interest. When $b \geq r$, the American call will never be exercised early, and valuation of c_{Eu} applies.

In equation (6.5.20), it is worthwhile to note that the early exercise premium of the American call option C_{Am} on a commodity approaches zero as the time to expiration of the option approaches zero. As $T \to 0$, $N[d_1(S^*)] \to 1$ and $\left\{1 - e^{(b-r)T}N[d_1(S^*)]\right\} \to 0$, $A_2 \to 0$ and thus $A_2(S/S^*)^{q_2} \to 0$.

A similar procedure is used to price an American put option. The American put option is a summation of the European put option and the early exercise premium

$$P_{Am}(S,\tau) = p_{Eu}(S,\tau) + \epsilon_P. \tag{6.5.22}$$

When the asset price is very large, $S \to \infty$, the American put early exercise premium is $\epsilon_P \to 0$. The positive q_2 in the $a_2S^{q_2}$ term creates the opposite and necessitates that $a_2 = 0$. Therefore, $f = a_1S^{q_1}$and

$$P_{Am}(S,\tau) = p_{Eu}(S,\tau) + ha_1S^{q_1}. \tag{6.5.23}$$

Below the critical asset price S^{**}, and the first constraint is that the early exercise premium plus the European put should equal the early exercise proceeds

$$K - S^{**} = p_{Eu}(S^{**},\tau) + hS^{**q_1}. \tag{6.5.24}$$

The quadratic approximation for an American put is

$$\begin{aligned} P_{Am}(S,\tau) &= p_{Eu}(S,\tau) + A_1(S/S^{**})^{q_1}, \qquad S > S^{**}, \\ P_{Am}(S,\tau) &= K - S, \qquad S \le S^{**}, \end{aligned} \tag{6.5.25}$$

where

$$A_1 = -(\frac{S^{**}}{q_1})(1 - e^{(b-r)T}\Phi[-d_1(S^{**})]). \tag{6.5.26}$$

An American put will always possess an early exercise premium as A_1 is always positive.

6.5.2. Critical Asset Price

In the previous section, only one step, the determination of the critical asset price S^* is not straightforward. We are interested in minimizing

$$g(S_i,\tau) = c_{Eu}(S^*,\tau) + \frac{1 - e^{(b-r)T}\Phi[d_1(S_i)]}{q_2}S^* - S^* + K \tag{6.5.27}$$

Barone-Addesi and Whaley (1987) implemented a modified Newton-Raphson search procedure to find the critical asset price.

The Newton-Raphson method finds the slope (the tangent line) of the function at $(x_0, g(x_0))$ and uses the zero of the tangent line as the next reference point. The process is repeated until the root is found or the convergence criteria are met;

$$S_{i+1} = S_i - \frac{g(S_i)}{g^{'}(S_i)}, \tag{6.5.28}$$

with

$$g^{'}(S_i,\tau) = (e^{(b-r)T}\Phi[d_1(S_i)])\left(1 - \frac{1}{q_2}\right) + \left(\frac{1}{q_2}\right)\left(1 - \frac{e^{(b-r)T}\Phi[d_1(S_i)]}{\sigma\sqrt{\tau}}\right) - 1. \tag{6.5.29}$$

The Newton-Raphson method is much more efficient than other "simple" methods such as the bisection method. However, the Newton-Raphson method requires the calculation of the derivative of a function at the reference point, which is not always easy. Furthermore, the tangent line often shoots wildly and might occasionally be trapped in a loop. The promised efficiency is then unfortunately too good to be true. It is recommended to monitor the step obtained by the Newton-Raphson method. When the step is too large or the value is oscillating, other more conservative methods should be considered.

6.5.3. Futures Option Quadratic Approximation

Exercising an option on a precious metal or a non-dividend paying stock does not generate cash flow unless it is sold. In contrast, exercising a futures option generates a cash flow of $F - S$ as the future is mark to market (daily, monthly etc.). As the futures option would only be exercised early if $F > S$, then the profit $F - S$ is immediately invested (bank account, bond etc.) to yield a profit $e^{r\tau}(F - K)$.

Applying the BAW model with $b = 0$ for a futures call option

$$\begin{gathered} C_{Am}(F,\tau) = c_{Eu}(F,\tau) + A_1(F/F^*)^{q_1}, \qquad F < F^*, \\ C_{Am}(F,\tau) = F - K, \qquad F \geqslant F^*, \end{gathered} \tag{6.5.30}$$

where

$$A_1 = (\frac{F^*}{q_1})(1 - e^{-rT}\Phi[d_1(S^*)]). \tag{6.5.31}$$

The future put option is

$$\begin{gathered} P_{Am}(F,\tau) = p_{Eu}(F,\tau) + A_2(F/F^*)^{q_2}, \qquad F > F^{**}, \\ P_{Am}(F,\tau) = K - F, \qquad F \leqslant F^{**}, \end{gathered} \tag{6.5.32}$$

where

$$A_2 = -(\frac{F^{**}}{q_2})(1 - e^{-rT}\Phi[-d_1(S^{**})]). \tag{6.5.33}$$

American futures call option

Option parameters	**Futures price**	**Crank-Nicolson FD method**	**Barone-Adesi Whaley**	**Black model**
$r = 0.08 \quad \sigma = 0.2 \quad T = 0.25$	80	0.0393	0.0407	0.0391
	90	0.6988	0.7052	0.6983
	100	3.9183	3.9353	3.9088
	110	10.8159	10.8268	10.7370
	120	20.0271	20.0214	19.7484
$r = 0.08 \quad \sigma = 0.4 \quad T = 0.5$	80	2.9755	3.0137	2.9618
	90	6.1991	6.2538	6.1597
	100	10.8963	10.9655	10.8053
	110	16.9666	17.0420	16.7828
	120	24.1884	24.2592	23.8570
$r = 0.03 \quad \sigma = 0.45 \quad T = 1$	80	7.6707	7.7485	7.6764
	90	12.0522	12.1091	11.9997
	100	17.3797	17.4348	17.2759
	110	23.5566	23.6127	23.3902
	120	30.4651	30.5185	30.2157

Table 6.1.: Theoritical American futures call option values using Crank-Nicolson Finite difference method in (5.2.8) on 200x200 grid, Barone-Adesi and Whaley quadratic approximation and Black's model (MATLAB **blkprice**). Exercise price $K = 100$.

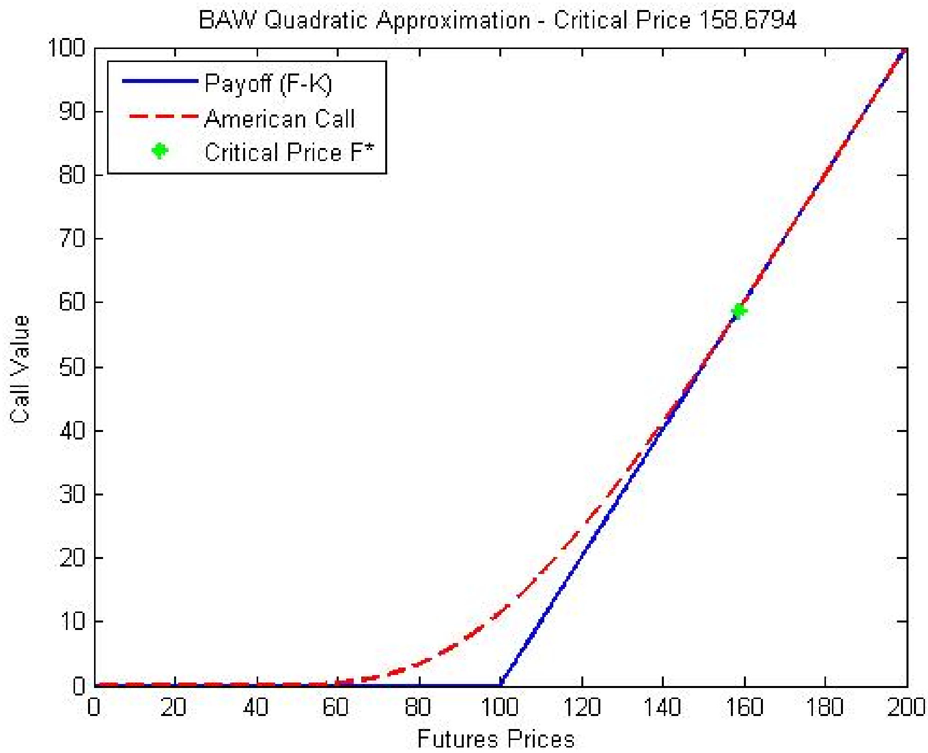


Figure 6.4.: Barone-Adesi and Whaley quadratic approximation of an American futures contract. Above the critical futures price, F^*, the American call is equal to the immediate exercise value $F - K$.

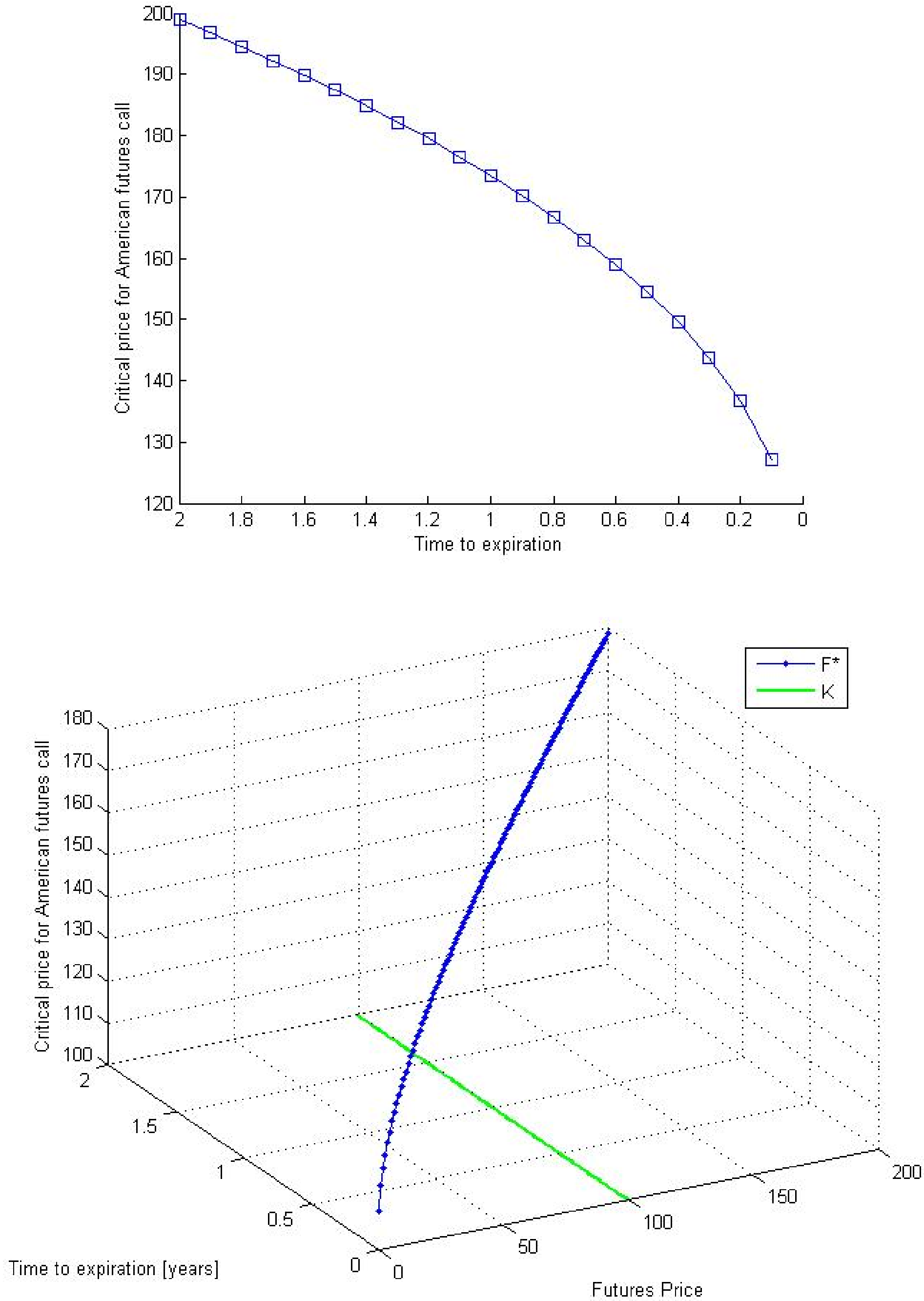


Figure 6.5.: The critical futures price, F^*, is a continuous decreasing function of time of the expiry timr τ. Here we used $r = 0.08$ and $\sigma = 0.2$

7. Appendix

A.1 Ito's Lemma

Ito's lemma is the chain rule for stochastic calculus. Let $x(t)$ follow the differential equation

$$\frac{dx}{dt} = a(x,t). \tag{7.0.1}$$

Now we consider a function of $x(t)$ and t. We call this function $f(x(t),t)$. Assuming that f is differentiable we can ask what the derivative of f is. To calculate it, we simply apply the chain rule

$$\frac{df(x,t)}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial t} = f_x \frac{dx}{dt} + f_t, \tag{7.0.2}$$

where we are using the notation $f_x = \frac{\partial f}{\partial x}$ and $f_t = \frac{\partial f}{\partial t}$. Finally, we can substitute in for $\frac{dx}{dt}$ from (7.0.1), giving

$$\frac{df(x,t)}{dt} = \frac{\partial f}{\partial x}a(x,t) + \frac{\partial f}{\partial t} = f_x a(x,t) + f_t. \tag{7.0.3}$$

This is a fairly straightforward calculation. However, when dealing with stochastic differential equations, the simple chain rule of ordinary calculus does not work. The reason is simple. Brownian motion is not differentiable so we can't really take its derivative or the derivative of any function of Brownian motion.

Ito's lemma for Brownian motion

Given the differential of $x(t)$, Ito's lemma allows us to compute the differential of a function of$x(t)$ and t. Hence, it is the "chain rule" for stochastic differential equations. The following result is Ito's lemma when $x(t)$ is a process governed by a stochastic differential equation driven by Brownian motion.

Lemma: Consider the stochastic differential equation (SDE)

$$dx = a(x,t)dt + b(x,t)dW, \tag{7.0.4}$$

and let $f(x, t)$ be a twice continuously differentiable function of x and t. Then

$$df(x,t) = (f_t + a(x,t)f_x + \frac{1}{2}b^2(x,t)f_{xx})dt + b(x,t)f_x dW. \tag{7.0.5}$$

Heuristic Proof: Consider writing the Taylor expansion of df

$$df = f(x(t+dt), t+dt) - f(x(t), t) = f_t dt + f_x dx + \frac{1}{2}f_{xx}(dx)^2 + f_{xt}dxdt + \dots \tag{7.0.6}$$

Now we substitute dx using $dx = adt + bdW$ which give

$$df = f_t dt + f_x(adt + bdW) + \frac{1}{2}f_{xx}(adt + bdW)^2 + f_{xt}(adt + bdW)dt + \dots$$
$$= f_t dt + f_x adt + f_x bdz + \frac{1}{2}f_{xx}(a^2dt^2 + 2abdtdW + b^2dW^2) + f_{xt}(adt^2 + bdWdt) + \dots$$

Now we take a crucial step, and only keep terms up to order dt using the following logic. The standard deviation of dW is of order $\sqrt{dt}$. Hence, we think of dz as being of order $dt^{1/2}$ and only keep terms up to order dt yielding

$$df = f_t dt + f_x adt + f_x bdW + \frac{1}{2}f_{xx}b^2dW^2 \dots \tag{7.0.7}$$

Finally we replace dW^2 by it's expectation dt which leads to Ito's lemma

$$df = (f_t + af_x a + \frac{1}{2}b^2 f_{xx})dt + bf_x dW. \tag{7.0.8}$$

Consequences for Stocks and Options

Suppose the stock price follows a geometric Brownian motion, hence $x(t) = S_t$, $a = \mu dt$, $b = \sigma S_t$. The value V of an option depends on S_t,$V = V(S_t, t)$. Assuming C^2-smoothness of V depending on S an dt, we apply Ito's lemma.

$$dV_t = \left(\frac{\partial V}{\partial S}\mu S_t + \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S_t^2\right)dt + \frac{\partial V}{\partial S}\sigma S_t dW \tag{7.0.9}$$

This SDE is used to derive the Black-Scholes PDE equation.

A.2 Derivation of Greeks

In this appendix, we derive formulas for $\Delta-$delta and $\Gamma-$gamma. The relatively lengthy derivation is for delta.

Derivation of Delta

The Black-Scholes formula for a plain vanilla European call expiration T, strike, K, is given by

$$c(S_t,t) = S_t \int_{-\infty}^{\frac{log\frac{S_t}{K}+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$$
$$-r^{-r(T-t)} K \int_{-\infty}^{\frac{log\frac{S_t}{K}+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du.$$

Rearrange and let $x_t = \frac{S-t}{Ke^{-r(T-t)}}$, giving

$$c(S_t,t) = Ke^{-r(T-t)} \left[\int_{-\infty}^{\frac{logx_t+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du - \int_{-\infty}^{\frac{logx_t-\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du \right]. \tag{7.0.10}$$

Now differentiate with respect to x_t

$$\frac{dc(x_t,t)}{dx_t} = Ke^{-r(T-t)} \left[\int_{-\infty}^{\frac{logx_t+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du \right]$$
$$+\frac{1}{\sigma\sqrt{T-t}} \left[\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{logx_t+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)^2} \right]$$
$$-\left[\frac{1}{x_t\sigma\sqrt{T-t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{logx_t-\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)^2} \right]$$

The last two terms in this expression sum to zero. To see this, on the right-hand side, use the substitution: $\frac{1}{x_t} = e^{-logx_t}$and then rearrange the exponent in the exponential function. Thus, we are left with

$$\frac{\partial c(x_t,t)}{\partial x_t} = Ke^{-r(T-t)} \left[\int_{-\infty}^{\frac{logx_t+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du \right] \tag{7.0.11}$$

Using the chain rule we can obtain

$$\frac{\partial c(x_t,t)}{\partial S_t} = \left[\int_{-\infty}^{\frac{log x_t + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du\right] = \Phi(d_1). \tag{7.0.12}$$

Derivation of Gamma

Once delta of a European call is obtained, the gamma will be the derivative of the delta. This gives

$$\frac{\partial^2 c(x_t,t)}{\partial S_t^2} = \frac{1}{S_t\sigma\sqrt{T-t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{log x_t + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)^2}. \tag{7.0.13}$$

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